15.083: Integer Programming and Combinatorial Optimization
Problem Set 6 Solutions

Problem (9.2(b)) Recall that Gomory cuts are obtained from fractional rows in the final simplex tableau and have the form $\sum_{j\in N}(\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor)x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$.

Problem (9.4) We will obtain the integer hull by performing successive Chvatal closures. We end up needing only the first Chvatal closure. We obtain the Chvatal closure by finding a TDI description for P with integral rhs and rounding down the lhs. The minimal faces for our inItial description of P are $(0,0,0)$, $(0,0,8/3)$, $(0,4,0)$, and $(8,0,0)$. We will now check that for each minimal face, we have tight constraints who's rhs define an integral generating wet for their respective cones:

1. $(0,0,0)$: the tight constraints $(-1,0,0),(0,-1,0),(0,0,-1)$ are unit vectors in their respective orthants so clearly define an integral basis.
2. $(0,0,8/3)$: here we need to add the generator $(0,0,1)$ with the tight constraint $x_3 \leq 8/3$ item $(0,4,0)$: here we need to add the generator $(0,1,0)$ with the tight constraint $x_2 \leq 4$
3. $(8,0,0)$: the tight constraints $(0,-1,0),(0,0,-1),(1,2,3)$ already form an integral basis.

Rounding down the rhs of the constraint $x_3 \leq 8/3 \Rightarrow x_3 \leq 2$ we obtain a polyhedron with all integral vertices, giving us P_I.

Problem (9.8)

(a) Consider any minimal face of the stable set polyhedron and the constraints $A_F x = b_F$ that are binding. The stable set polyhedron is known to have all vertices with entries $0, 1/2, 1$ where entries of $1/2$ appear only around cycles. Thus we can decompose A_F into a block diagonal matrix $\begin{pmatrix} A_V & 0 & 0 & \cdots \\ 0 & A_{C_1} & 0 & \cdots \\ 0 & 0 & \ddots & 0 \end{pmatrix}$. Where A_V is the node-edge incidence matrix of a bipartite graph (nodes with value 0 in one partite, value 1 in the other) plus some unit vectors and is thus unimodular; and therefore TDI in any system. The matrices A_{C_j} correspond to stability constraints along disjoint cycles. The only integral points in $y \in cone((a_{C_j})_i)$ that we could not generate from integer multiples of $(a_{C_j})_i$ are of the form $y_i = k$ for $i \in C, k$ *odd* for some cycle C (since we can only generate $y_i = 2$ by adding the tight inequalities around a cycle). Thus adding the valid constraints $\sum_{e\in C} x_e \leq \frac{|C|}{2}$ $\forall cycles\ C$ completes the integral generating set and leaves us with a TDI system. Rounding down the rhs of this system then gives us P_1. But the inequalities $\sum_{e\in C} x_e \leq \lfloor \frac{|C|}{2} \rfloor$ $\forall cycles\ C$ are valid for $P_{1/2}$ so $P_{1/2} = P_1$.

(b) We need only show that $P_{1/2} = conv(\mathfrak{F})$. Assume $P_{1/2}$ is not integral. Then there exists some c for which the unique maximum of cx over $P_{1/2}$ contains a fractional entry x_i. We have $x_j > 0 \Rightarrow c_j > 0$ since we could otherwise decrease x_j, remain feasible and have the same objective. Since x_i is fractional, we have $c_i > 0$ and either $x_{i-1} + x_i = 1$ or $x_i + x_{i+1} = 1$, otherwise we could increase x_i. wlog assume $x_{i-1} + x_i = 1$. Suppose that all variables with indices from $\underline{i}$ to $\bar{i}$ are fractional and satisfy $x_i + x_{i-1} = 1$ $i = \underline{i} + 1 \ldots \bar{i}$. We claim that either $x_{\bar{i}} + x_{\bar{i}+1} = 1$ or $x_{\underline{i}} + x_{\underline{i}-1} = 1$; for if not $x_{\bar{i}+1} = x_{\underline{i}-1} = 0$, and we can shift weight between our fractional variables while remaining feasible and attaining an objective value no worse than x, contradicting the unique minimality of x. Therefore we can continue to grow our set of fractional variables until we show that all x_i are fractional and $x_i + x_{i-1} = 1$, $x_n + x_1 = 1$. Adding up these equalities we have $2\sum_i x_i = |C|$ which violates the odd-cycle inequality. Thus $P_{1/2}$ can have no fractional vertices.

Problem (11.3) Note that the optimal integer objective is 1 since n is odd. In any branch and bound node, let S_0 be the set of indices from $1, ..., n$ for which the corresponding variable is set to 0, and S_1 be the set of indices for which the corresponding variable is set to 1. This yields the following LP relaxation:

$$\min x_{n+1} : x_i \in [0,1] \ x_{n+1} + \sum_{i \notin S_0 \cup S_1} 2x_i = n - 2|S_1|$$

As long as $|S_0|, |S_1| \leq \frac{n-1}{2}$, the optimal value of this relaxation is 0. Since $n - 2|S_1|$ is odd, any optimal solution must contain at least one fractional component and we can continue branching. We will continue branching until we reach infeasible nodes where $|S_0|$ or $|S_1|$ are strictly greater than $\frac{n-1}{2}$ or when $|S_0| = |S_1| = \frac{n-1}{2}$ and we obtain the optimal integer solution with objective 0. When we obtain the optimal solution in some node, we cannot prove optimality through LP bounds until we exhaust all other feasible branches (since they will have a lower bound of 0. Since any node with less than $\frac{n-1}{2}$ set values is feasible, we must explore at least $2^{\frac{n-1}{2}}$ nodes.

Problem (11.5) Let x_t be the on-hand inventory at the beginning of period t and y_t be the amount ordered in period t. The system dynamics are then $y_{t+1} = y_t + x_t - d_t$. The terminal boundary condition is $V_{T+1}(x) = 0$. The DP equation is then given by:

$$V_t(x_t) = \min_{y_t \geq 0, y_t \geq x_t - d_t} \mathbf{1}\{y_t > 0\}c_t + y_t p_t + h_t(x_t + y_t - d_t) + V_{t+1}(x_t + y_t - d_t)$$

The optimal value is then $V_1(x_1)$ where x_1 is our initial on-hand inventory.

Problem (11.12(a)) Notice that we can construct any tour in the neighborhood by starting at node j and examining each successive node in our initial tour ordering, adding it to either the beginning or end of our tour. We then consider a graph on $n^2 + 1$ nodes labeled (i, k) indicating that node i is at the begging of our partially constructed tour and node k is at the end with a dummy node (j', j') corresponding to a completed tour. From each node (i, k) there is an arc to (w, k) and (i, w) indicating that we add node w to the beginning or end of our tour respectively with costs $c_{w,i}$ and $c_{k,w}$ respectively. The node (j', j') is connected to all nodes (i, k) corresponding to hamiltonian paths with cost (i, k). So we have an acyclic graph on n^2 nodes with $m = n^2 + n$ edges. Solving the shortest path problem from (j, j) to (j', j') gives us the shortest tour in the neighborhood of interest. Since we have an acyclic graph, we can solve the shortest path problem in $O(m) = O(n^2)$ time with the reaching algorithm.

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