15.083J/6.859J Integer Optimization

Lecture 10: Solving Relaxations

1 Outline

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- The key geometric result behind the ellipsoid method
- The ellipsoid method for the feasibility problem
- The ellipsoid method for optimization
- Problems with exponentially many constraints

2 The Ellipsoid method

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- $\boldsymbol{D}$ is an $n \times n$ positive definite symmetric matrix
- A set E of vectors in $\Re^n$ of the form

$$E = E(\boldsymbol{z}, \boldsymbol{D}) = \left\{ \boldsymbol{x} \in \Re^n \mid (\boldsymbol{x} - \boldsymbol{z})' \boldsymbol{D}^{-1} (\boldsymbol{x} - \boldsymbol{z}) \leq 1 \right\}$$

  is called an **ellipsoid** with center $\boldsymbol{z} \in \Re^n$

2.1 The algorithm intuitively

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- Problem: Decide whether a given polyhedron

$$P = \left\{ \boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \right\}$$

  is nonempty

- Key property: We can find a new ellipsoid E_{t+1} that covers the half-ellipsoid and whose volume is only a fraction of the volume of the previous ellipsoid E_t

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2.2 Key Theorem

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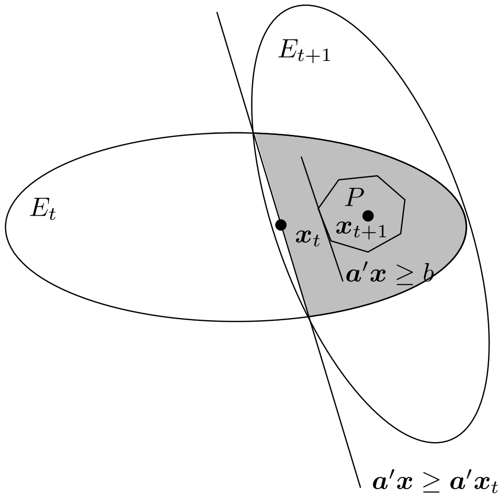
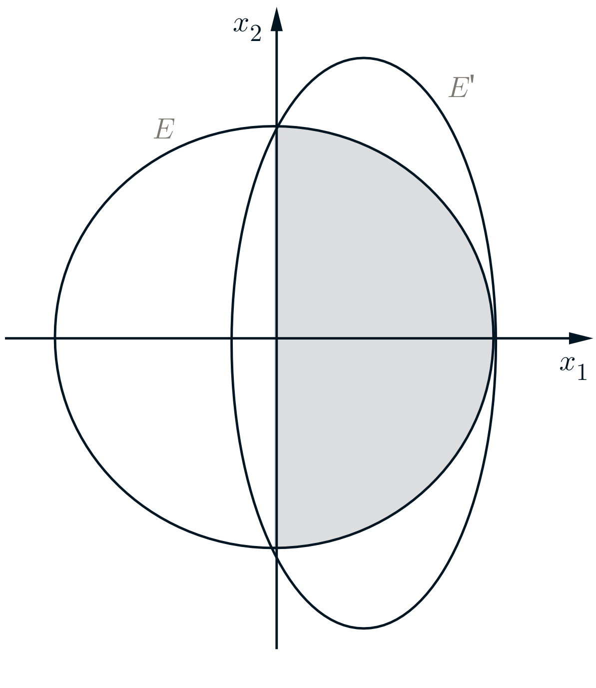
- $E = E(\boldsymbol{z}, \boldsymbol{D})$ be an ellipsoid in $\Re^n$; $\boldsymbol{a}$ nonzero n-vector.
- $H = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{a}'\boldsymbol{x} \geq \boldsymbol{a}'\boldsymbol{z}\}$

$$\begin{aligned} \overline{\boldsymbol{z}} &= \boldsymbol{z} + \frac{1}{n+1} \frac{\boldsymbol{D}\boldsymbol{a}}{\sqrt{\boldsymbol{a}'\boldsymbol{D}\boldsymbol{a}}}, \\ \overline{\boldsymbol{D}} &= \frac{n^2}{n^2 - 1} \left(\boldsymbol{D} - \frac{2}{n+1} \frac{\boldsymbol{D}\boldsymbol{a}\boldsymbol{a}'\boldsymbol{D}}{\boldsymbol{a}'\boldsymbol{D}\boldsymbol{a}} \right). \end{aligned}$$

- The matrix $\overline{\boldsymbol{D}}$ is symmetric and positive definite and thus $E' = E(\overline{\boldsymbol{z}}, \overline{\boldsymbol{D}})$ is an ellipsoid
- $E \cap H \subset E'$
- $\mathrm{Vol}(E') < e^{-1/(2(n+1))} \, \mathrm{Vol}(E)$



E_{t+1}
E_t
P
$\boldsymbol{x}_t$
$\boldsymbol{x}_{t+1}$
$\boldsymbol{a}'\boldsymbol{x} \geq b$
$\boldsymbol{a}'\boldsymbol{x} \geq \boldsymbol{a}'\boldsymbol{x}_t$



x_2
E'
E
x_1

2.3 Illustration

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2.4 Assumptions

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- A polyhedron P is **full-dimensional** if it has positive volume
- The polyhedron P is bounded: there exists a ball $E_0 = E(\boldsymbol{x}_0, r^2\boldsymbol{I})$, with volume V, that contains P
- Either P is empty, or P has positive volume, i.e., $\text{Vol}(P) > v$ for some $v > 0$
- E_0, v, V, are a priori known
- We can make our calculations in infinite precision; square roots can be computed exactly in unit time

2.5 Input-Output

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Input:

- A matrix $\boldsymbol{A}$ and a vector $\boldsymbol{b}$ that define the polyhedron $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{a}_i'\boldsymbol{x} \geq b_i,\ i = 1, \ldots, m\}$
- A number v, such that either P is empty or $\text{Vol}(P) > v$
- A ball $E_0 = E(\boldsymbol{x}_0, r^2\boldsymbol{I})$ with volume at most V, such that $P \subset E_0$

Output: A feasible point $\boldsymbol{x}^* \in P$ if P is nonempty, or a statement that P is empty

2.6 The algorithm

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1. (Initialization)
Let $t^* = \lceil 2(n+1)\log(V/v) \rceil$; $E_0 = E(\boldsymbol{x}_0, r^2\boldsymbol{I})$; $\boldsymbol{D}_0 = r^2\boldsymbol{I}$; $t = 0$.

2. (Main iteration)

- If $t = t^*$ stop; P is empty.
- If $\boldsymbol{x}_t \in P$ stop; P is nonempty.
- If $\boldsymbol{x}_t \notin P$ find a violated constraint, that is, find an i such that $\boldsymbol{a}_i'\boldsymbol{x}_t < b_i$.
- Let $H_t = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{a}_i'\boldsymbol{x} \geq \boldsymbol{a}_i'\boldsymbol{x}_t\}$. Find an ellipsoid E_{t+1} containing $E_t \cap H_t$: $E_{t+1} = E(\boldsymbol{x}_{t+1}, \boldsymbol{D}_{t+1})$ with

$$\begin{aligned}\boldsymbol{x}_{t+1} &= \boldsymbol{x}_t + \frac{1}{n+1}\frac{\boldsymbol{D}_t\boldsymbol{a}_i}{\sqrt{\boldsymbol{a}_i'\boldsymbol{D}_t\boldsymbol{a}_i}},\\ \boldsymbol{D}_{t+1} &= \frac{n^2}{n^2-1}\left(\boldsymbol{D}_t - \frac{2}{n+1}\frac{\boldsymbol{D}_t\boldsymbol{a}_i\boldsymbol{a}_i'\boldsymbol{D}_t}{\boldsymbol{a}_i'\boldsymbol{D}_t\boldsymbol{a}_i}\right).\end{aligned}$$

- $t := t + 1$.

2.7 Correctness

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Theorem: Let P be a bounded polyhedron that is either empty or full-dimensional and for which the prior information $\boldsymbol{x}_0$, r, v, V is available. Then, the ellipsoid method decides correctly whether P is nonempty or not, i.e., if $\boldsymbol{x}_{t^*-1} \notin P$, then P is empty

2.8 Proof

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- If $\boldsymbol{x}_t \in P$ for $t < t^*$, then the algorithm correctly decides that P is nonempty
- Suppose $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{t^*-1} \notin P$. We will show that P is empty.
- We prove by induction on k that $P \subset E_k$ for $k = 0, 1, \ldots, t^*$. Note that $P \subset E_0$, by the assumptions of the algorithm, and this starts the induction.

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- Suppose $P \subset E_k$ for some $k < t^*$. Since $\boldsymbol{x}_k \notin P$, there exists a violated inequality: $\boldsymbol{a}'_{i(k)}\boldsymbol{x} \geq \boldsymbol{b}_{i(k)}$ be a violated inequality, i.e., $\boldsymbol{a}'_{i(k)}\boldsymbol{x}_k < \boldsymbol{b}_{i(k)}$, where $\boldsymbol{x}_k$ is the center of the ellipsoid E_k
- For any $\boldsymbol{x} \in P$, we have

$$\boldsymbol{a}'_{i(k)}\boldsymbol{x} \geq \boldsymbol{b}_{i(k)} > \boldsymbol{a}'_{i(k)}\boldsymbol{x}_k$$

- Hence, $P \subset H_k = \left\{\boldsymbol{x} \in \Re^n \mid \boldsymbol{a}'_{i(k)}\boldsymbol{x} \geq \boldsymbol{a}'_{i(k)}\boldsymbol{x}_k\right\}$
- Therefore, $P \subset E_k \cap H_k$

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By key geometric property, $E_k \cap H_k \subset E_{k+1}$; hence $P \subset E_{k+1}$ and the induction is complete

$$\frac{\mathrm{Vol}(E_{t+1})}{\mathrm{Vol}(E_t)} < e^{-1/(2(n+1))}$$

$$\frac{\mathrm{Vol}(E_{t^*})}{\mathrm{Vol}(E_0)} < e^{-t^*/(2(n+1))}$$

$$\mathrm{Vol}(E_{t^*}) < Ve^{-\lceil 2(n+1)\log\frac{V}{v}\rceil/(2(n+1))} \leq Ve^{-\log\frac{V}{v}} = v$$

If the ellipsoid method has not terminated after t^* iterations, then $\mathrm{Vol}(P) \leq \mathrm{Vol}(E_{t^*}) \leq v$. This implies that P is empty

2.9 Binary Search

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- $P = \left\{x \in \Re \mid x \geq 0, x \geq 1, x \leq 2, x \leq 3\right\}$
- $E_0 = [0, 5]$, centered at $x_0 = 2.5$

- Since $x_0 \notin P$, the algorithm chooses the violated inequality $x \leq 2$ and constructs E_1 that contains the interval $E_0 \cap \{x \mid x \leq 2.5\} = [0, 2.5]$
- The ellipsoid E_1 is the interval $[0, 2.5]$ itself
- Its center $x_1 = 1.25$ belongs to P
- This is binary search

2.10 Boundedness of P

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Let $\boldsymbol{A}$ be an $m \times n$ integer matrix and let $\boldsymbol{b}$ a vector in $\Re^n$. Let U be the largest absolute value of the entries in $\boldsymbol{A}$ and $\boldsymbol{b}$.
Every extreme point of the polyhedron $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$ satisfies

$$-(nU)^n \leq x_j \leq (nU)^n, \quad j = 1, \ldots, n$$

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- All extreme points of P are contained in

$$P_B = \left\{\boldsymbol{x} \in P \;\middle|\; |x_j| \leq (nU)^n,\ j = 1, \ldots, n\right\}$$

- Since $P_B \subseteq E\big(\boldsymbol{0}, n(nU)^{2n}\boldsymbol{I}\big)$, we can start the ellipsoid method with $E_0 = E\big(\boldsymbol{0}, n(nU)^{2n}\boldsymbol{I}\big)$
-

$$Vol(E_0) \leq V = \big(2n(nU)^n\big)^n = (2n)^n(nU)^{n^2}$$

2.11 Full-dimensionality

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Let $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$. We assume that $\boldsymbol{A}$ and $\boldsymbol{b}$ have integer entries, which are bounded in absolute value by U. Let

$$\epsilon = \frac{1}{2(n+1)}\big((n+1)U\big)^{-(n+1)}.$$

Let

$$P_\epsilon = \big\{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} - \epsilon\boldsymbol{e}\big\},$$

where $\boldsymbol{e} = (1, 1, \ldots, 1)$.

(a) If P is empty, then P_ϵ is empty.

(b) If P is nonempty, then P_ϵ is full-dimensional.

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Let $P = \big\{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\big\}$ be a full-dimensional bounded polyhedron, where the entries of $\boldsymbol{A}$ and $\boldsymbol{b}$ are integer and have absolute value bounded by U. Then,

$$\mathrm{Vol}(P) > v = n^{-n}(nU)^{-n^2(n+1)}$$

2.12 Complexity

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- $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$, where $\boldsymbol{A}$, $\boldsymbol{b}$ have integer entries with magnitude bounded by some U and has full rank. If P is bounded and either empty or full-dimensional, the ellipsoid method decides if P is empty in $O\big(n \log(V/v)\big)$ iterations
- $v = n^{-n}(nU)^{-n^2(n+1)}, \qquad V = (2n)^n(nU)^{n^2}$
- Number of iterations $O\big(n^4 \log(nU)\big)$

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- If P is arbitrary, we first form P_B, then perturb P_B to form $P_{B,\epsilon}$ and apply the ellipsoid method to $P_{B,\epsilon}$
- Number of iterations is $O\big(n^6 \log(nU)\big)$.
- It has been shown that only $O(n^3 \log U)$ binary digits of precision are needed, and the numbers computed during the algorithm have polynomially bounded size
- The linear programming feasibility problem with integer data can be solved in polynomial time

3 The ellipsoid method for optimization

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$$\begin{array}{rl} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \end{array} \qquad\qquad \begin{array}{rlcl} \max & \boldsymbol{b}'\boldsymbol{\pi} & & \\ \text{s.t.} & \boldsymbol{A}'\boldsymbol{\pi} & = & \boldsymbol{c} \\ & \boldsymbol{\pi} & \geq & \boldsymbol{0}. \end{array}$$

By strong duality, both problems have optimal solutions if and only if the following system of linear inequalities is feasible:

$$\boldsymbol{b}'\boldsymbol{p} = \boldsymbol{c}'\boldsymbol{x}, \qquad \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \qquad \boldsymbol{A}'\boldsymbol{p} = \boldsymbol{c}, \qquad \boldsymbol{p} \geq \boldsymbol{0}.$$

LO with integer data can be solved in polynomial time.

3.1 Sliding objective

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- We first run the ellipsoid method to find a feasible solution $\boldsymbol{x}_0 \in P = \big\{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\big\}$.
- We apply the ellipsoid method to decide whether the set
$$P \cap \big\{\boldsymbol{x} \in \Re^n \mid \boldsymbol{c}'\boldsymbol{x} < \boldsymbol{c}'\boldsymbol{x}_0\big\}$$
is empty.
- If it is empty, then $\boldsymbol{x}_0$ is optimal. If it is nonempty, we find a new solution $\boldsymbol{x}_1$ in P with objective function value strictly smaller than $\boldsymbol{c}'\boldsymbol{x}_0$.

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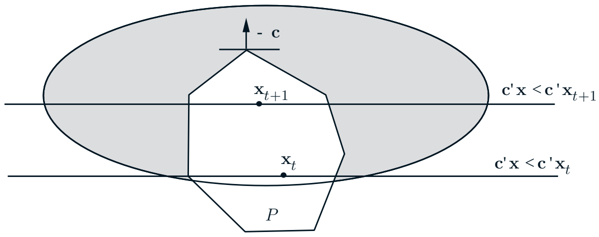
- More generally, every time a better feasible solution $\boldsymbol{x}_t$ is found, we take $P \cap \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{c}'\boldsymbol{x} < \boldsymbol{c}'\boldsymbol{x}_t\}$ as the new set of inequalities and reapply the ellipsoid method.



3.2 Performance in practice

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- Very slow convergence, close to the worst case
- Contrast with simplex method
- The ellipsoid method is a tool for classifying the complexity of linear programming problems

4 Problems

4.1 Example

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$$\min \sum_i c_i x_i$$

$$\sum_{i \in S} a_i x_i \geq |S|, \qquad \text{for all subsets } S \text{ of } \{1, \dots, n\}$$

- There are 2^n constraints, but are described concisely in terms of the n scalar parameters $a_1, \dots, a_n$
- Question: Suppose we apply the ellipsoid algorithm. Is it polynomial?
- In what?

4.2 The input

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- Consider $\min \boldsymbol{c'x}$ s.t. $\boldsymbol{x} \in P$
- P belongs to a family of polyhedra of special structure
- A typical polyhedron is described by specifying the dimension n and an integer vector $\boldsymbol{h}$ of *primary data*, of dimension $O(n^k)$, where $k \geq 1$ is some constant.
- In example, $\boldsymbol{h} = (a_1, \dots, a_n)$ and $k = 1$
- U_0 be the largest entry of $\boldsymbol{h}$

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- Given n and $\boldsymbol{h}$, P is described as $\boldsymbol{Ax} \geq \boldsymbol{b}$
- $\boldsymbol{A}$ has an arbitrary number of rows
- U largest entry in $\boldsymbol{A}$ and $\boldsymbol{b}$. We assume

$$\log U \leq Cn^{\ell} \log^{\ell} U_0$$

5 The separation problem

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Given a polyhedron $P \subset \Re^n$ and a vector $\boldsymbol{x} \in \Re^n$, the **separation problem** is to:

- Either decide that $\boldsymbol{x} \in P$, or
- Find a vector $\boldsymbol{d}$ such that $\boldsymbol{d}'\boldsymbol{x} < \boldsymbol{d}'\boldsymbol{y}$ for all $\boldsymbol{y} \in P$

What is the separation problem for

$$\sum_{i \in S} a_i x_i \geq |S|, \qquad \text{for all subsets } S \text{ of } \{1, \ldots, n\}?$$

6 Polynomial solvability

6.1 Theorem

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If we can solve the separation problem (for a family of polyhedra) in time polynomial in n and $\log U$, then we can also solve linear optimization problems in time polynomial in n and $\log U$. If $\log U \leq Cn^{\ell} \log^{\ell} U_0$, then it is also polynomial in $\log U_0$

- Proof ?
- Converse is also true
- Separation and optimization are polynomially equivalent

6.2 MST

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$$\begin{aligned} IZ_{MST} = \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq 1 \qquad \forall\, S \subseteq V, S \neq \emptyset, V \\ & \sum_{e \in E} x_e = n - 1 \\ & x_e \in \{0, 1\}. \end{aligned}$$

How can you solve the LP relaxation?

6.3 TSP

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$$x_e = \begin{cases} 1, & \text{if edge } e \text{ is included in the tour.} \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq 2, \quad S \subseteq E \\ & \sum_{e \in \delta(i)} x_e = 2, \quad i \in V \\ & x_e \in \{0,1\} \end{aligned}$$

How can you solve the LP relaxation?

6.4 Probability Theory

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- Events A_1, A_2
- $P(A_1) = 0.5,\ P(A_2) = 0.7,\ P(A_1 \cap A_2) \leq 0.1$
- Are these beliefs consistent?
- General problem: Given n events A_i $i \in N = \{1, \ldots, n\}$, beliefs

$$\mathrm{P}(A_i) \leq p_i, \qquad i \in N,$$

$$\mathrm{P}(A_i \cap A_j) \geq p_{ij}, \qquad i, j \in N,\ i < j.$$

- Given the numbers p_i and p_{ij}, which are between 0 and 1, are these beliefs consistent?

6.4.1 Formulation

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$$x(S) = \mathrm{P}\Big(\big(\cap_{i \in S} A_i\big) \cap \big(\cap_{i \notin S} \overline{A}_i\big)\Big),$$

$$\begin{aligned} \sum_{\{S \mid i \in S\}} x(S) &\leq p_i, && i \in N, \\ \sum_{\{S \mid i,j \in S\}} x(S) &\geq p_{ij}, && i, j \in N,\ i < j, \\ \sum_{S} x(S) &= 1, \\ x(S) &\geq 0, && \forall\ S. \end{aligned}$$

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The previous LP is feasible if and only if there does not exist a vector $(\boldsymbol{u}, \boldsymbol{y}, z)$ such that

$$\begin{aligned} \sum_{i,j \in S, i<j} y_{ij} + \sum_{i \in S} u_i + z &\geq 0, && \forall\ S, \\ \sum_{i,j \in N, i<j} p_{ij} y_{ij} + \sum_{i \in N} p_i u_i + z &\leq -1, \\ y_{ij} \leq 0,\ u_i \geq 0, & && i, j \in N,\ i < j. \end{aligned}$$

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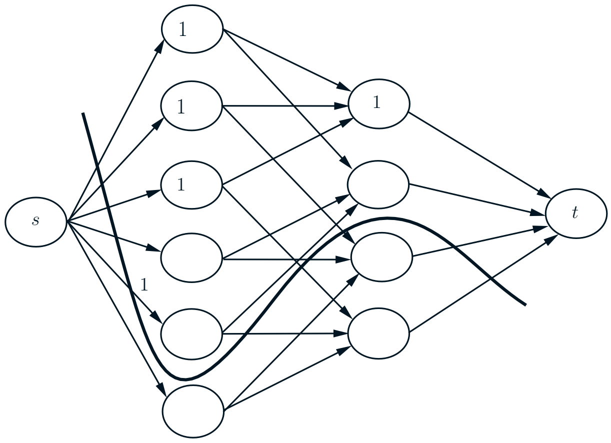
Separation problem:

$$z^* + \min_S f(S) = \sum_{i,j \in S, i<j} y^*_{ij} + \sum_{i \in S} u^*_i \geq 0?$$

Example: $y^*_{12} = -2$, $y^*_{13} = -4$, $y^*_{14} = -4$, $y^*_{23} = -4$, $y^*_{24} = -1$, $y^*_{34} = -7$, $u^*_1 = 9$, $u^*_2 = 6$, $u^*_3 = 4$, $u^*_4 = 2$, and $z^* = 2$ Slide 36



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- The minimum cut corresponds to $S_0 = \{3, 4\}$ with value $c(S_0) = 21$.
- $f(S_0) = \sum_{i,j \in S_0, i<j} y^*_{ij} + \sum_{i \in S_0} u^*_i = -7 + 4 + 2 = -1$
- $f(S) + z^* \geq f(S_0) + z^* = -1 + 2 = 1 > 0, \qquad \forall\ S$
- Given solution $(\boldsymbol{y}^*, \boldsymbol{u}^*, z^*)$ is feasible

7 Conclusions

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- Ellipsoid algorithm can characterize the complexity of solving LOPs with an exponential number of constraints
- For practical purposes use dual simplex
- Ellipsoid method is an important theoretical development, not a practical one