15.083J/6.859J Integer Optimization

Lecture 7: Ideal formulations III

1 Outline

Slide 1

- Minimal counterexample
- Lift and project

2 Matching polyhedron

Slide 2

$$P_{\text{matching}} = \left\{ \boldsymbol{x} \;\middle|\; \begin{array}{l} \sum_{e \in \delta(\{i\})} x_e = 1,\ i \in V, \\ \sum_{e \in \delta(S)} x_e \geq 1,\ S \subset V, |S| \text{ odd},\ |S| \geq 3, \\ 0 \leq x_e \leq 1,\ e \in E \end{array} \right\}.$$

- F set of perfect matchings in G.
- Theorem: For the perfect matching problem

$$P_{\text{matching}} = \text{conv}(F).$$

2.1 Proof Outline

Slide 3

- $\text{conv}(F) \subset P_{\text{matching}}$.
- For reverse: Assume $G = (V, E)$ is a graph such that $P_{\text{matching}} \not\subset \text{conv}(F)$, and $|V| + |E|$ is the smallest.
- $\boldsymbol{x}$ be an extreme point of P_{matching} not in $\text{conv}(F)$.
- For each edge $e = \{u, v\}$, $x_e > 0$, otherwise we could delete e from E.
- $x_e < 1$, otherwise we could replace V by $V \setminus \{u, v\}$ and E by all edges in E incident to $V \setminus \{u, v\}$.
- $|E| > |V|$; otherwise, either G is disconnected (in this case one of the components of G will be a smaller counterexample), or G has a node of degree one (in this case the edge e incident to v satisfies $x_e = 1$), or G is the disjoint union of cycles (in this case the theorem holds trivially).
- $\boldsymbol{x}$ extreme point of P_{matching}, there are $|E|$ linearly independent tight constraint.
- There exists a $S \subset V$ with $|S|$ odd, $|S| \geq 3$, $|V \setminus S| \geq 3$, and

$$\sum_{e \in \delta(S)} x_e = 1.$$

- Contract $V \setminus S$ to a single new node u, to obtain $G' = (S \cup \{u\}, E')$.
- $x'_e = x_e$ for all $e \in E(S)$, and for $v \in S$,

$$x'_{\{u,v\}} = \sum_{\{j \in V \setminus S, \{v,j\} \in E\}} x_{\{v,j\}}.$$

$\boldsymbol{x}'$ satisfies constraints with respect to G'.

- As G is a smallest counterexample, $\boldsymbol{x}'$ belongs to the convex hull of matchings on G',

$$\boldsymbol{x}' = \sum_{M'} \lambda_{M'} \boldsymbol{\chi}^{M'}.$$

- Contract S to a single new node t we obtain a graph $G'' = ((V \setminus S) \cup \{t\}, E'')$ and a vector $\boldsymbol{x}''$:

$$\boldsymbol{x}'' = \sum_{M''} \mu_{M''} \boldsymbol{\chi}^{M''}.$$

- "Glue together" perfect matchings M' and M''

$$\boldsymbol{x} = \sum_{e \in \delta(S)} \sum_{M \text{ perfect matching: } M \cap \delta(S) = \{e\}} \frac{\lambda_{M'} \mu_{M''}}{x_e} \boldsymbol{\chi}^M$$

3 Lift and project

SLIDE 4

- $S = \{\boldsymbol{x} \in \mathcal{Z}^n \mid \boldsymbol{Ax} \leq \boldsymbol{b}\}$.
- **(Lift)** Multiply $\boldsymbol{Ax} \leq \boldsymbol{b}$ by x_j and $1 - x_j$

$$\begin{aligned} (\boldsymbol{Ax})x_j &\leq \boldsymbol{b}x_j \qquad (*) \\ (\boldsymbol{Ax})(1 - x_j) &\leq \boldsymbol{b}(1 - x_j) \end{aligned}$$

  and substitute $y_{ij} = x_i x_j$ for $i, j = 1, \ldots, n,\ i \neq j$ and $x_j = x_j^2$. Let $L_j(P)$ be the resulting polyhedron.
- **(Project)** Project $L_j(P)$ back to the $\boldsymbol{x}$ variables by eliminating variables $\boldsymbol{y}$. Let P_j be the resulting polyhedron, i.e., $P_j = (L_j(P))_x$.

3.1 Theorem

SLIDE 5

$$P_j = \text{conv}(P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j \in \{0, 1\}\})$$

Proof:

- $\boldsymbol{x}' \in P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j \in \{0, 1\}\}$ and $y'_{ij} = x'_i x'_j$.
- Since $x'_j = (x'_j)^2$ and $\boldsymbol{Ax}' \leq \boldsymbol{b}$, $(\boldsymbol{x}', \boldsymbol{y}') \in L_j(P)$ and thus $\boldsymbol{x}' \in P_j$. Hence,

$$\text{conv}(P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j \in \{0, 1\}\}) \subseteq P_j.$$

- If $P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j = 0\} = \emptyset$, then from the Farkas lemma there exists $\boldsymbol{u} \geq \boldsymbol{0}$, such that $\boldsymbol{u}'\boldsymbol{A} = -\boldsymbol{e}_j$ and $\boldsymbol{u}'\boldsymbol{b} = -1$. Thus, for all $\boldsymbol{x}$ satisfying (*) we have

$$\boldsymbol{u}'\boldsymbol{Ax}(1 - x_j) \leq \boldsymbol{u}'\boldsymbol{b}(1 - x_j).$$

  Hence, for all $\boldsymbol{x} \in P_j$

$$-\boldsymbol{e}'_j \boldsymbol{x}(1 - x_j) = -x_j(1 - x_j) \leq -(1 - x_j).$$

  Replacing x_j^2 by x_j, we obtain that $x_j \geq 1$ is valid for P_j. Since, in addition, $P_j \subseteq P$, we conclude that

$$P_j \subseteq P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j = 1\} = \text{conv}(P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j \in \{0, 1\}\}).$$

- Similarly, if $P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j = 1\} = \emptyset$, then

$$P_j \subseteq \text{conv}(P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j \in \{0,1\}\}).$$

- Suppose $P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j = 0\} \neq \emptyset$, $P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j = 1\} \neq \emptyset$.
- We prove that all valid inequalities for $\text{conv}(P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j \in \{0,1\}\})$ are also valid for P_j.
- $\boldsymbol{a}'\boldsymbol{x} \leq \alpha$ a valid inequality for $\text{conv}(P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j \in \{0,1\}\})$.
- $\boldsymbol{x} \in P$. If $x_j = 0$, then for all $\lambda \in \mathcal{R}$ $\boldsymbol{a}'\boldsymbol{x} + \lambda x_j = \boldsymbol{a}'\boldsymbol{x} \leq \alpha$.
- If $x_j > 0$, then there exists $\lambda \leq 0$, such that for all $\boldsymbol{x} \in P$,

$$\boldsymbol{a}'\boldsymbol{x} + \lambda x_j \leq \alpha.$$

- Analogously, since $\boldsymbol{a}'\boldsymbol{x} \leq \alpha$ is valid for $P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j = 1\}$, there exists some $\nu \leq 0$ such that for all $\boldsymbol{x} \in P$,

$$\boldsymbol{a}'\boldsymbol{x} + \nu(1 - x_j) \leq \alpha.$$

- For all $\boldsymbol{x}$ satisfying (*),

$$\begin{aligned} (1 - x_j)(\boldsymbol{a}'\boldsymbol{x} + \lambda x_j) &\leq (1 - x_j)\alpha \\ x_j(\boldsymbol{a}'\boldsymbol{x} + \nu(1 - x_j)) &\leq x_j\alpha. \end{aligned}$$

- Hence,

$$\boldsymbol{a}'\boldsymbol{x} + (\lambda + \nu)(x_j - x_j^2) \leq \alpha.$$

- After setting $x_j^2 = x_j$ we obtain that for all $\boldsymbol{x} \in P_j$, $\boldsymbol{a}'\boldsymbol{x} \leq \alpha$, thus all valid inequalities for $\text{conv}(P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j \in \{0,1\}\})$ are also valid for P_j, and thus $P_j \subseteq \text{conv}(P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_j \in \{0,1\}\})$.

3.2 Example

Slide 6

$P = \{(x_1, x_2)' \mid 2x_1 - x_2 \geq 0, 2x_1 + x_2 \leq 2, x_1 \geq 0, x_2 \geq 0\}$.

$$\begin{aligned} 2x_1^2 - x_1x_2 &\geq 0 \\ 2x_1(1 - x_1) - x_2(1 - x_1) &\geq 0 \\ 2x_1^2 + x_1x_2 &\leq 2x_1 \\ 2x_1(1 - x_1) + x_2(1 - x_1) &\leq 2(1 - x_1) \\ x_1^2 &\geq 0 \\ x_1(1 - x_1) &\geq 0 \\ x_2x_1 &\geq 0 \\ x_2(1 - x_1) &\geq 0. \end{aligned}$$

$y = x_1 x_2$, $x_1^2 = x_1$

$$\begin{aligned} 2x_1 - y &\geq 0 \\ -x_2 + y &\geq 0 \\ y &\leq 0 \\ x_2 - y &\leq 2 - 2x_1 \\ x_1 &\geq 0 \\ 0 &\geq 0 \\ y &\geq 0 \\ x_2 - y &\geq 0. \end{aligned}$$

This implies that $y = 0$,

$$\begin{aligned} x_1 &\geq 0 \\ -x_2 &\geq 0 \\ x_2 &\leq 2 - 2x_1 \\ x_1 &\geq 0 \\ x_2 &\geq 0, \end{aligned}$$

which leads to

$$\begin{aligned} P_1 &= \{(x_1, x_2)' \mid 0 \leq x_1 \leq 1,\ x_2 = 0\} \\ &= \mathrm{conv}(P \cap \{(x_1, x_2)' \mid x_1 \in \{0, 1\}\}). \end{aligned}$$

3.3 Convex hull

SLIDE 7

- $P_{i_1, i_2, \ldots, i_t} = ((P_{i_1})_{i_2} \ldots)_{i_t}$.
- Theorem: The polyhedron $P_{i_1, i_2, \ldots, i_t}$ satisfies:

$$P_{i_1, \ldots, i_t} = \mathrm{conv}(P \cap \{\boldsymbol{x} \in \mathcal{R}^n \mid x_i \in \{0, 1\},\ i \in \{i_1, \ldots, i_t\}\}).$$

- $P_{1, \ldots, n} = P_I$.