15.083: Integer Programming and Combinatorial Optimization
Final Exam Solutions

Problem (1)

(a) F

(b) F

(c) F

(d) T

(e) F

(f) F

(g) F

(h) F

(i) T

Problem (2)

(a) For $x \in C$, by the non-negativity of x and $\frac{1}{\lambda}$ we have

$$\sum_{i=1}^{n} \lfloor \frac{a_i}{\lambda} \rfloor x_i \leq \sum_{i=1}^{n} \frac{a_i}{\lambda} x_i \leq \frac{b}{\lambda}$$

Since x is restricted to take integer values and each lhs coefficient is integral, we can round down the rhs to obtain the desired inequality.

(b) We write the inequality by $fx \leq b$. By (a), this inequality is valid for C. $\lambda < b < \lambda n \Rightarrow 1 \leq \lfloor \frac{b}{\lambda} \rfloor \leq n-1$. Let F be the face induced by $fx = g$. Clearly all integer points with $k = \lfloor \frac{b}{\lambda} \rfloor$ components equal to 1 (the rest equal to zero) are in F. Let $hx = d$ be any equality that holds on F. Consider the points $x^1, x^2 \in F$ with

$$\begin{aligned} x^1 &= (1, 1, \cdots, 1, 1, 0, 0, \cdots, 0) \\ x^2 &= (1, 1, \cdots, 1, 0, 1, 0, \cdots, 0) \end{aligned}$$

both with k components equal to 1. We then have $hx^1 - hx^2 = 0 \Rightarrow h_k = h_{k-1}$. Extending this argument, we can obtain $h_i = h_j$ for all pairs i, j. Thus $h = \alpha f$ for some α and by theorem A.2 in the book $fx \leq b$ must be facet defining.

Problem (3)

(a) We define decision variables $x \in \{0,1\}^{|A|}$ such that $x_e = 1$ if $e \in B$, 0 otherwise. We define the sets $\delta^+(v), \delta^-(v)$ to be the set of arcs entering/leaving node v respectively. We can the define the following binary optimization model for our problem:

$$\boxed{\begin{array}{l} \max \sum_{e \in A} w_e x_e \\ \text{subject to} \\ \begin{array}{rcll} x_e + x_{e'} & \leq & 1 & \forall v \in V, e \in \delta^+(v), e' \in \delta^-(v) \\ x_e & \in & \{0,1\} & \forall e \in A \end{array} \end{array}}$$

(b) This polyhedron is not in general integral. Consider the graph one i, j, k with arcs $(i, j), (j, k), (k, i)$. The solution $(1/2, 1/2, 1/2)$ is a vertex since it cannot be written as a convex combination of feasible integer vectors.

(c) If the graph is bipartite, then we can divide the edges into two partites as follows : one partite corresponding to arcs with heads in the first node partite, the other partite corresponding to arcs with heads in the second node partite). Then the constraint matrix is the node-edge incidence matrix of an undirected bipartite graph which is TU, so the polyhedron is integral.

(d) Consider a directed odd cycle C in the graph. Adding up all the constraints corresponding to the edges in the cycle and the nodes the visit yields: $\sum_{a \in C} x_a \leq \frac{|C|}{2}$. So the inequality $\sum_{a \in C} x_a \leq \frac{|C|-1}{2}$ is valid for the integer hull but not the relaxation.

(e) We can solve the separation problem over the odd-cycle inequalities in polynomial time and have a polynomial number of head-tail constraints (at most one for each pair of arcs) which can be checked in polynomial time. Therefore we can solve the separation problem over our new polytope in polynomial time and use ellipsoid to solve the optimization in polynomial time.

(f) Not integral in general. As discussed in part (g) which follows, our polytope is equivalent to the stable set polytope with some of the cycle inequalities included for a dual graph which is not in general integral. The directed cycle inequalities do not capture all cycle inequalities in this dual graph. For instance take the graph on 5 nodes with arcs $(1, 2), (3, 1), (4, 3), (3, 5), (5, 1)$.

(g) It is a stable set relaxation polytope on a "dual" undirected graph where we have a node for each arc in the primal with an edge between them if the tail of one is the head of the other.

Problem (4)

(a) Let $G = (V, E)$ be an instance of the Steiner tree problem, consider the complete graph G' on V and define the cost of an edge $\{u, v\}$ in G' to be the cost of a shortest u-v-path in G. Note that we may calculate the shortest u-v-path for each pair in polynomial time, so we have a polynomial time transformation of G to G'. Since the cost of each arc in G' is a shortest path cost in G, we have that G' satisfies the triangle inequality, and thus is an instance of the metric Seiner tree problem on a complete graph. Note that any solution T for G is feasible for G' and the realization in G' must have cost no greater than the realization in G (since the cost of each arc $\{u, v\} \in E$ is replaced by the shortest path cost in G'). Thus the cost of the optimal tree for G' cannot exceed that for G. Given an optimal solution T' to this new metric Steiner tree problem, we can obtain a Steiner tree on G with no greater cost by replacing each edge $\{u, v\}$ in T' by the shortest u-v path calculated in G; this may lead to extra edges in the tree which we can simply delete to obtain a tree with no greater cost.

(b) If we traverse the optimal Steiner tree twice, we obtain a Euler tour visiting all nodes in R. We can transform this Euler tour into a hamiltonian tour with no greater cost by considering a hamiltonian tour in the order the nodes are visiting on the Steiner 2-tour since our graph is metric. Thus $TSP \leq 2MStT$, and $MST \leq TSP$ yielding the desired result.

(c) Consider an instance with n required vertices $i = 1, ..., n$ and one Steiner vertex $n + 1$ with $c_{i,j} = 2$ $1 \leq i, j \leq n$ and $c_{i,n+1} = 1$ $1 \leq i \leq n$. The cost on the minimum spanning tree (MST) is $2 * (n - 1)$ whereas the cost of the minimum Steiner tree (MStT) is n. Thus $\frac{MST}{MStT} = \frac{2n-1}{n} \rightarrow_{n \rightarrow \infty} 2$.

Problem (5)

(a)

$$
\begin{array}{l}
\min \sum_{t=1}^{T} (f_t x_t + p_t y_t + h_t s_t) \\
\text{subject to} \\
\begin{array}{rcll}
s_{t-1} + y_t - d_t & = & s_t & t = 1, ..., T \\
s_t & \geq & 0 & t = 1, ..., T-1 \\
s_0, s_T & = & 0 & \\
y_t & \leq & \left(\sum_{s=t}^{T} d_s\right) x_t & t = 1, ..., T \\
x_t & \in & \{0, 1\} & t = 1, ..., T \\
y_t & \geq 0 & t = 1, ..., T &
\end{array}
\end{array}
$$

(b) If $\sum_{i \in C} x_i = 0$ then the inequality is clearly valid since $x_i = 0 \Rightarrow y_i = 0$. So we assume $\sum_{i \in C} x_i > 0$. We prove by induction on k. If $k = 0$ then we have $s_0 \geq 0$ which is valid. Suppose the inequality is valid for $\bar{k} = 0, ..., k-1$. We have two cases:

(i) $k \notin C$: Then

$$\begin{aligned}
\sum_{i \in C} y_i &\leq \sum_{i \in C} \left(\sum_{t=i}^{k-1} d_t \right) x_i + s_{k-1} \\
&= \sum_{i \in C} \left(\sum_{t=i}^{k-1} d_t \right) x_i + s_k + d_k - y_k \\
&\leq \sum_{i \in C} \left(\sum_{t=i}^{k-1} d_t \right) x_i + s_k + d_k
\end{aligned}$$

Since at least 1 of the $x_i = 1$ for $i \in C$, we can pull the existing d_k term into that summation and add the d_k term to all others to obtain:

$$\sum_{i \in C} y_i \leq \sum_{i \in C} \left(\sum_{t=i}^{k} d_t \right) x_i + s_k$$

(ii) $k \in C$: Then

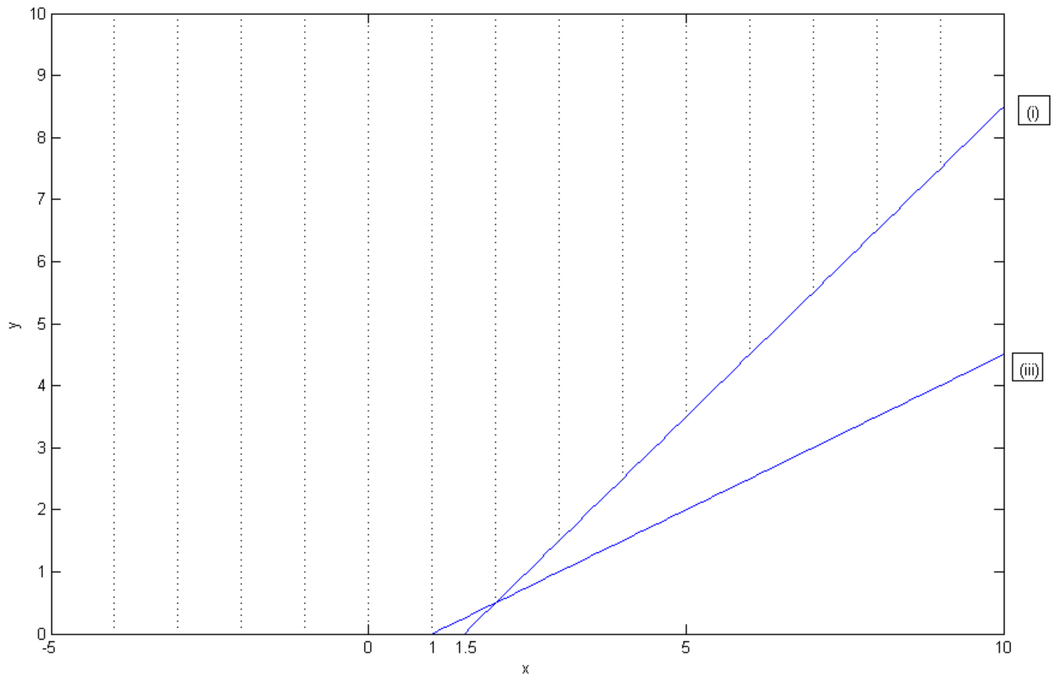
$$\begin{aligned}
\sum_{i \in C} y_i &\leq \sum_{i \in C \setminus \{k\}} \left(\sum_{t=i}^{k-1} d_t \right) x_i + s_{k-1} + y_k \\
&= \sum_{i \in C \setminus \{k\}} \left(\sum_{t=i}^{k-1} d_t \right) x_i + s_k + d_k
\end{aligned}$$

If $x_k = 1$ then we can add a d_k term to each summation and pull the existing d_k term in by changing the range of summation from $C \setminus \{k\}$ to C to obtain the desired inequality. If $x_k = 0$ then as in (i), there is an $i \in C \setminus \{k\}$ for which $x_i = 1$ and we can pull the existing d_k term into that summation and add d_k to all others to obtain the desired inequality.

Problem (6)

(a)

(i) For $b = \frac{3}{2}$:



(ii) We consider the sets $S_1 = S \cup \{x : x \le \lfloor b \rfloor$ and $S_2 = S \cup \{x : x \ge \lfloor b \rfloor + 1$. For S_1, we multiply $x \le \lfloor b \rfloor$ by $(1 - f_0)$ and add $0 \le y$ to obtain $(1 - f_0)(x - \lfloor b \rfloor) \le y$ as a valid inequality. For S_2 we multiply $x \ge \lfloor b \rfloor + 1$ by f_0 to obtain $-f_0 \ge -f_0(x - \lfloor b \rfloor)$ and add $b \ge x - y$ (valid for $S_1 \subseteq S$) to obtain

$$\begin{aligned} -f_0 + b &\ge -f_0(x - \lfloor b \rfloor) + x - y \\ \Rightarrow & \\ y &\ge f_0 - b - f_0 x + f_0 \lfloor b \rfloor + x \\ &= (1 - f_0)(x - \lfloor b \rfloor) \end{aligned}$$

We then have $(1 - f_0)(x - \lfloor b \rfloor) \le y$ is valid for $S_1 \cup S_2 = S$ which rearranges to the desired inequality.

(iii) see (i) above

(b)

(i) Here we simply take the inequality $ax + gy \le b$ and round down the lhs coefficients for $x_j : f_j \le f_0$ (which we can do since x is non-negative), and drop the non-negative term $\sum_{j:g_j \ge 0} g_j y_j$. We can write this algebraically as:

$$\begin{aligned} & \textstyle\sum_{j:f_j \le f_0} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} a_j x_j + \sum_{j:g_j < 0} g_j y_j \\ = \;& \textstyle\sum_{j:f_j \le f_0} (a_j - f_j) x_j + \sum_{j:f_j > f_0} a_j x_j + \sum_{j:g_j < 0} g_j y_j \\ = \;& \textstyle\sum_{j=1}^n a_j x_j - \underbrace{\sum_{j:f_j \le f_0} f_j x_j}_{\ge 0} + \sum_{j:g_j < 0} g_j y_j \\ \le \;& \textstyle\sum_{j=1}^n a_j x_j + \sum_{j:g_j < 0} g_j y_j \\ \le \;& \textstyle\sum_{j=1}^n a_j x_j + \sum_{j=1}^n g_j y_j \\ \underbrace{\le}_{(x,y) \in P} \;& b \end{aligned}$$

(ii)

$$
\begin{aligned}
w - z &= \sum_{j:f_j \le f_0} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} (\lceil a_j \rceil) x_j + \sum_{j:g_j<0} g_j y_j - \sum_{j:f_j>f_0} (1 - a_j + \lfloor a_j \rfloor) x_j \\
&= \sum_{j:f_j \le f_0} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} (\lfloor a_j \rfloor + 1) x_j + \sum_{j:g_j<0} g_j y_j - \sum_{j:f_j>f_0} (1 - a_j + \lfloor a_j \rfloor) x_j \\
&= \sum_{j:f_j \le f_0} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} a_j x_j + \sum_{j:g_j<0} g_j y_j \\
&\underbrace{\le}_{(i)} b
\end{aligned}
$$

(iii) By (b)(ii) we have that $(x, y) \in conv(T)$ such that there exist a mappings w, z as defined which satisfy (w, z) in this polytope. Thus by (a)(ii):

$$
\begin{aligned}
\lfloor b \rfloor &\ge w - \frac{1}{1-f_0} z \\
&= \sum_{j:f_j \le f_0} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} (\lceil a_j \rceil) x_j - \frac{1}{1-f_0} \left(- \sum_{j:g_j<0} g_j y_j + \sum_{j:f_j>f_0} (1 - f_j) x_j \right) \\
&= \sum_{j:f_j \le f_0} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} (\lfloor a_j \rfloor + 1) x_j - \frac{1}{1-f_0} \left(- \sum_{j:g_j<0} g_j y_j + \sum_{j:f_j>f_0} (1 - f_j) x_j \right) \\
&= \sum_{j=1}^{n} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} x_j + \frac{1}{1-f_0} \sum_{j:g_j<0} g_j y_j - \sum_{j:f_j>f_0} \frac{1-f_j}{1-f_0} x_j \\
&= \sum_{j=1}^{n} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} \left(1 - \frac{1-f_j}{1-f_0} \right) x_j + \frac{1}{1-f_0} \sum_{j:g_j<0} g_j y_j \\
&= \sum_{j=1}^{n} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} \left(\frac{f_j - f_0}{1-f_0} \right) x_j + \frac{1}{1-f_0} \sum_{j:g_j<0} g_j y_j \\
&= \sum_{j=1}^{n} \lfloor a_j \rfloor x_j + \sum_{j=1}^{n} \left(\frac{(f_j - f_0)^+}{1-f_0} \right) x_j + \frac{1}{1-f_0} \sum_{j:g_j<0} g_j y_j
\end{aligned}
$$

(c) Adding non-negative slack to obtain $ax + gy + s = b$ is equivalent to adding a new continuous variable $y_{p+1} := s$, $g_{p+1} := 1$ and examining the equality $\sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{p+1} g_j y_j$. The associated GMI cut is:

$$
\sum_{j:f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{j:f_j > f_0} \frac{1-f_j}{1-f_0} x_j + \sum_{j \in \{1,\dots,n\}:g_j>0} \frac{g_j}{f_0} y_j - \sum_{j:g_j<0} \frac{g_j}{1-f_0} y_j + \frac{1}{f_0} s \ge 1
$$

Plugging in $s = b - \sum_{j=1}^{n} a_j x_j - \sum_{j=1}^{p} g_j y_j$ and multiplying through by f_0 we obtain:

$$
\sum_{j=1}^{n} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} \left(f_j - f_0 \frac{1-f_j}{1-f_0} \right) x_j + \left(1 + \frac{f_0}{1-f_0} \right) \sum_{j:g_j<0} g_j y_j \le b - f_0 = \lfloor b \rfloor
$$

which is equivalent to:

$$
\sum_{j=1}^{n} \lfloor a_j \rfloor x_j + \underbrace{\sum_{j:f_j > f_0} \frac{f_j - f_0}{1-f_0} x_j}_{= \sum_{j=1}^{n} \frac{(f_j - f_0)^+}{1-f_0} x_j} + \left(\frac{1}{1-f_0} \right) \sum_{j:g_j<0} g_j y_j \le b - f_0 = \lfloor b \rfloor
$$