15.083: Integer Programming and Combinatorial Optimization
Problem Set 1 Solutions

Due 9/16/2009

Problem (1.2) Let $x_i = 1$ if we pick player i, 0 otherwise

$$
\boxed{
\begin{array}{l}
\max\limits_{x} \sum_{i=1}^{20} s_i x_i \\
\text{subject to} \\
\begin{array}{rcl}
\sum_{i=1}^{5} x_i & \geq & 3 \\
\sum_{i=4}^{11} x_i & \geq & 4 \\
\sum_{i=9}^{16} x_i & \geq & 4 \\
\sum_{i=16}^{20} x_i & \geq & 3 \\
\sum_{i \in \{4,8,15,20\}} x_i & \geq & 2 \\
\sum_{i=1}^{20} x_i & = & 12 \\
\sum_{i=1}^{20} r_i x_i & \geq & 12r \\
\sum_{i=1}^{20} a_i x_i & \geq & 12a \\
\sum_{i=1}^{20} s_i x_i & \geq & 12s \\
\sum_{i=1}^{20} h_i x_i & \geq & 12h \\
\sum_{i=1}^{20} d_i x_i & \geq & 12d \\
\sum_{i \in \{5,9\}} x_i & \leq & 1 \\
x_2 - x_{19} & = & 0 \\
\sum_{i \in \{1,7,12,16\}} x_i & \leq & 3
\end{array}
\end{array}
}
$$

Problem (1.7)

(a) If the LP below is feasible and its optimal value is greater than zero, then it is possible to separate the points by class.

$$\begin{array}{|l|}\hline \max_{c,z} z \\ \text{subject to} \\ \begin{array}{rcll} c'x_i & \leq & 1 & \forall i : a_i = 0 \\ c'x_i - z & \geq & 1 & \forall i : a_i = 1 \end{array} \\ \hline \end{array}$$

(b) For M sufficiently large and ϵ sufficiently small:

$$\begin{array}{|l|}\hline \max_{c,u_1,u_2,z,\beta_1,\beta_2} \sum_i z_i \\ \text{subject to} \\ \begin{array}{rcll} c'x_i - 1 & \leq & Mu_{2i} & \forall i \\ 1 - c'x_i + \epsilon & \leq & Mu_{1i} & \forall i \\ u_{1i} + u_{2i} & = & 1 & \forall i \\ y_i - B_1'x_i & \leq & z_i + Mu_{2i} & \forall i \\ -y_i + B_1'x_i & \leq & z_i + Mu_{2i} & \forall i \\ y_i - B_2'x_i & \leq & z_i + Mu_{1i} & \forall i \\ -y_i + B_2'x_i & \leq & z_i + Mu_{1i} & \forall i \\ u_{1i}, u_{2i} \in \{0,1\} & & \forall i & \end{array} \\ \hline \end{array}$$

Problem (1.21)

(a) We will show that $\mathcal{F}$ and $\mathcal{F}'$ are both the set of incidence vectors of Directed Hamiltonian Cycles. Let y be the incidence vector of a Directed Hamiltonian Cycle. It is easy to check that $y \in \mathcal{F}$. Now we will derive a vector u such that (u, y) satisfies the constraints for $\mathcal{F}'$. Starting with node 1, travel along the tour induced by y visiting each node in the graph. For each node i visited, set u_i equal to its position in the tour. For instance, if the Hamiltonian cycle is $1 \to 4 \to 2 \to 3 \to 1$, we set $u_4 = 2, u_2 = 3, u_3 = 4$. We then have that the maximum entry of u is n and the minimum entry of u is 2. Therefore for any $i, j | y_{ij} = 0 : u_i - u_j + ny_{ij} \leq n - 2 < n - 1$. And for any $i, j | y_{ij} = 1 : u_j = u_i + 1 \to u_i - u_j + ny_{ij} = n - 1$. Thus $y \in \mathcal{F}'$.
Now let $y \in \{y : \sum_{i|(i,j) \in A} y_{ij} = 1, \sum_{j|(i,j) \in A} y_{ij} = 1\}$ be the incidence vector of an edge-set that is not a Directed Hamiltonian cycle. By these "conservation of flow" constraints the digraph induced by y must contain a Directed cycle that is not Hamiltonian. Let $|C|$ be the nodes set of nodes visited in this cycle. We have $\sum_{(i,j) \in A | i \in C, j \notin C} y_{ij} = 0$; so $y \notin \mathcal{F}$. Now for each edge connecting nodes $i, j \in C$ let us sum the constraints $u_i - u_j + ny_{ij} \leq n - 1$. We have $\sum_{(i,j) \in E(S)} u_i - u_j + ny_{ij} = n|C| \leq (n-1)|C| = \sum_{(i,j) \in E(S)} n - 1$ which is a contradiction. Thus $y \notin \mathcal{F}'$.

(b) Let $y \in P_{tsp-dcut}$. We wish to show that $\exists u : u_i - u_j + ny_{ij} \leq n - 1$. This is true if and only if the following LP is feasible:

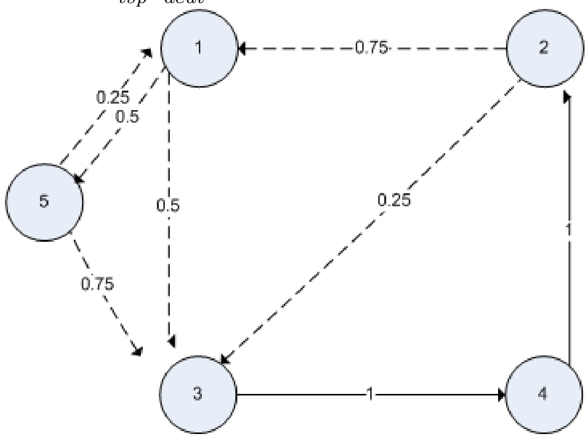
$$\begin{array}{|l|}\hline \max_u 0 \\ \text{subject to} \\ \begin{array}{rcll} u_i - u_j & \leq & n - 1 - ny_{ij} & \forall (i,j) \in A, i, j \neq 1 \end{array} \\ \hline \end{array}$$

The above LP is feasible if and only if its dual is bounded:

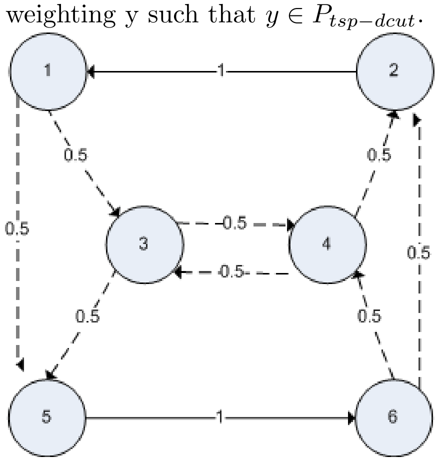
$$\begin{array}{|l|}\hline \min_v \sum_{(i,j) \in A, i,j \neq 1} (n - 1 - ny_{ij}) v_{ij} \\ \text{subject to} \\ \begin{array}{rcll} \sum_j v_{ij} - \sum_j v_{ji} & = & 0 & \forall i \in G, i \neq 1 \end{array} \\ \hline \end{array}$$

Let $P = \{v : \sum_j v_{ij} - \sum_j v_{ji} = 0 \quad \forall i \in G, i \neq 1\}$ be the feasible set of this dual.
Thus it is equivalent to show that $\forall v \in P$ we have $\sum_{(i,j) \in A, i,j \neq 1} (n - 1 - ny_{ij}) v_{ij} \geq 0$. Since P is a cone, it is sufficient to check this condition for all its extreme rays. Notice that the description of P is comprised of directed conservation of flow constraints. It is not difficult to see that any point $v \in P$ can be written as a conic combination of incidence vectors of directed cycles on the nodes $2, ..., n$ and thus these cycles are the extreme rays of P.
We then have $y \in P_{tsp-polynomial}$ if for all directed cycles C on $2, ..., n$ $(n-1)|C| - ny(C) \geq 0$ or, rearranging terms: $y(C) \leq |C| - \frac{|C|}{n}$. But since $y \in P_{tsp-dcut}$ we have

$\sum_{(i,j)\in A|i\in C, j\notin C} y_{ij} \geq 1 \rightarrow y(C) \leq |C| - 1$ which is a tighter condition. Therefore $y \in P_{tsp-polynomial}$.
That the inclusion is strict can be seen in the following example of an edge-graph which is in $P_{tsp-polynomial}$ but not $P_{tsp-dcut}$.



(c) Consider the dcut formulation of TSP over a complete graph on 6 nodes. The figure below shows an edge weighting y such that $y \in P_{tsp-dcut}$.



Now suppose that y can be written as a convex combination of K Directed Hamiltonian Cycles: $y = \sum_{k=1}^{K} \lambda_k y^k; \sum_{k=1}^{K} \lambda_k = 1; \lambda_k > 0 \forall k$. All such Cycles must then have $y_{56}^k = 1$ and $y_{21} = 1$. And for at least one such cycle $\bar{k}$ we must have $y_{43}^{\bar{k}} = 1$. However one the complete graph, there are only 2 Directed Hamiltonian Cycles for which $y_{21}^k = y_{56}^k = y_{43}^k = 1$: $1 \rightarrow 4 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 2 \rightarrow 1$ and $1 \rightarrow 5 \rightarrow 6 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1$. But $y_{14} = y_{32} = 0$ so y cannot be written as a convex combination of either of these vectors and any other Hamiltonian Cycles. Thus $y \notin conv(\mathcal{F})$.

Problem (1.23)

(a) AFL yields an optimal objective of 8352 whereas FL yields 13,245. So while AFL's is "better" in that it is lower, FL's provides a better bound on the objective of the integer problem.

(b) Both formulations have 4,020 variables that are upper and lower bounded. Not including variable bounds, FL has 4,020 constraints and AFL has 220 constraints. Thus AFL has smaller dimension.

(c) FL has found an integral solution, AFL has not.

(d) Even though FL takes longer to solve as an LP, the additional constraints provide a much tighter formulation cutting off a large volume of non-integral solutions. Thus in general, it will lead to better bounds on the integral objective and will be "more likely" to find an integral solution than AFL across instances.

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