# 15.083: Integer Programming and Combinatorial Optimization
# Problem Set 4 Solutions

Due 10/21/2009

**Problem (4.10)**
Let $Z_R, Z_{NR}$ be the optimal cost with and without release dates respectively. Let $NR$ be the set of acceptable completion times in the problem with no release dates. Release dates insert the constraints $C_i \geq p_i + r_i$ which we relax to obtain a problem with no release dates. For fixed $\lambda$, by prop 3.6 we have an optimal ordering (w.l.o.g $1, ..., n$) and thus can obtain the optimal cost in closed form:

$$\begin{aligned}
Z_R &\geq Z(\lambda) \\
&= \min_{C \in NR} \left\{ \sum_{i=1}^{n} w_i C_i + \sum_{i=1}^{n} \lambda_i \left( r_i + p_i - C_i \right) \right\} \\
&= \min_{C \in NR} \left\{ \sum_{i=1}^{n} (w_i - \lambda_i) C_i + \sum_{i=1}^{n} \lambda_i \left( r_i + p_i \right) \right\} \\
&= \sum_{i=1}^{n} (w_i - \lambda_i) \sum_{j=1}^{i} p_j + \sum_{i=1}^{n} \lambda_i \left( r_i + p_i \right) \\
&= \sum_{i=1}^{n} w_i \sum_{j=1}^{i} p_j + \sum_{i=1}^{n} \lambda_i \left( r_i - \sum_{j=1}^{i-1} p_j \right) \\
&\geq Z_{NR} + \sum_{i=1}^{n} \lambda_i \left( r_i - \sum_{j=1}^{i-1} p_j \right)
\end{aligned}$$

we wish to maximize this lower bound over $\lambda_i \in \{0, w_i\}$. Such a maximization yields an IP which is difficult to solve (since selecting specific values for $\lambda_i$ change the order of jobs. We can bound this IP from below in polynomial time in the following manner: We order the jobs such that $p_1 \geq p_2 \geq \cdots \geq p_n$, define the quantity $p_{i,k} = \sum_{j=1}^{i-1} p_j + \sum_{j=i+1}^{k} p_j$ which bounds from above the time it will take to process all jobs prior to i if we schedule job i to be the $k^{th}$ processed. Then define $a_{i,k} = w_i(r_i - p_{i,k})^+$. We may then solve the assignment problem:

$$\boxed{\begin{array}{l}
Z_{Assign} = \min \sum_{i,k} a_{i,k} x_{i,k} \\
\text{subject to} \\
\begin{array}{rcll}
\sum_i x_{i,k} &=& 1 & \forall k \\
\sum_k x_{i,k} &=& 1 & \forall i \\
x_{i,k} &\in& \{0,1\} & \forall i,k
\end{array}
\end{array}}$$

and bound $Z_R$ as follows:

$$Z_R \quad \geq \quad Z_{NR} + Z_{Assign}$$

**Problem (4.11)** Relaxing the constraint $x = y$ is equivalent to optimizing over the set $\{(x, y) \in conv(Ax \geq b, Dy \geq d) : x = y\} = \{(x, y) : x \in conv(Ax \geq b), y \in conv(Dy \geq d), x = y\} = \{(x, y) \in conv(Ax \geq b) \cap conv(Dx \geq d)\}$

**Problem (4.12)**

(a)

$$\boxed{\begin{array}{l} \max \displaystyle\sum_{(i,j)\in E} w_{ij}x_{ij} \\ \text{subject to} \\ \begin{array}{rcll} x_{st} & = & 1 & \\ x_{ij}+x_{jk}+x_{ik} & \leq & 2 & \forall i,j,k \\ x_{ij}-x_{jk}-x_{ik} & \leq & 0 & \forall i,j,k \\ x_{ij} & \in & \{0,1\} & \forall i,j,k \end{array} \end{array}}$$

(b) These inequalities state that for any 3 node, at least 2 lie in the same partition and if $i$ and $j$ are in different partitions then for some other node $k$ it must be that $k$ is in a different partition from either $i$ or $j$.

(c) We could relax the constraints $x_{ij}+x_{jk}+x_{ik} \leq 2$ and $x_{ij}-x_{jk}-x_{ik} \leq 0$ to end up with a subproblem that is efficaciously solvable. We could apply the subgradient algorithm (4.2) to optimize over our multipliers using subgradients of $2-x_{ij}-x_{jk}-x_{ik}$ and $-x_{ij}+x_{jk}+x_{ik}$ and an appropriately chosen stepsize. We need to be wary of the fact that our multipliers need to be kept non-negative, so at each step, we should round any negative multipliers up to 0. The simplex algorithm will have to cope with $O(n^2)$ variables and $O(n^3)$ constraints, but the constraint matrix is very sparse, whereas each of the Lagrangian subproblems have a simple closed form solution computable in $O(n^2)$ time. So it is difficult to judge which method will converge faster. Since the polyhedron we are optimizing over in the Lagrangian relaxation is integral, it will not produce a tighter bound.

**Problem (5.8)**

(a) Given a solution $x^*$ we first check that $0 \leq x^*_e \leq 1$. The we assign weights $x^*_e$ to each edge $e \in V$. For each edge $(i,j)$ we solve the min $i-j$ cut problem to obtain a set $S_{(i,j)}$ and check that $\sum_{e\in\delta(S_{(i,j)})} x^*_e \geq r_{(i,j)}$. If all of these constraints are satisfied then $x^*$ is feasible, else we generate a separating hyperplane.

(b) We can use the same approach as part (a) by letting $r_{(i,j)} = \mathbf{1}\{i,j \in T\}$