15.083: Integer Programming and Combinatorial Optimization
Midterm Exam

10/26/2009

Problem (1) (50 pts) Indicate whether the following italicized statements are true or false. Provide a supporting argument and/or short proof.

(a) [5 pts] Consider the problem max $\sum_{j=1}^{n} c_j|x_j|$ subject to $\sum_{j=1}^{n} a_j|x_j| \leq b$.
[T/F]: *The problem can be modeled as a linear integer optimization problem.*

(b) [5 pts] Let $(N, \mathcal{F})$ be a matroid with associated rank function $r(\cdot)$.
[T/F]: *For all pairs of sets $T_1, T_2 \subset N$ with $|T_1| = |T_2|$, we have $r(T_1) = r(T_2)$.*

(c) [5 pts] Let $x^* \in \mathbb{R}^n$ be an optimal solution to $Z_{LP} = \min_{Ay \leq b} c^T y$ and let $x \in \mathbb{Z}^n : Ax \leq b$ be a solution obtained from x^* by a randomized rounding procedure. Suppose $E[c'x] = c'x^* = Z_{LP}$.
[T/F]: *It is possible that $E\left[(c'x - Z_{LP})^2\right] > 0$.*

(d) [5 pts] Let $P = \{x \in \mathbb{R}^7 \mid Ax = b, f'x \geq d\}$ be a polyhderon with $rank(A) = 3$, in which the inequality $f'x \geq d$ defines a face of dimension 3.
[T/F]: *The inequality $f'x \geq d$ can be deleted from P.*

(e) [8 pts] Let $N = \{1, ..., n\}$. Consider the knapsack polytope $P_{KN} = conv\{x \in \{0,1\}^n : \sum_{i=1}^{n} w_i x_i \leq b\}$.

Suppose we identify a minimal cover $C \subseteq N$ with the following properties:

- $\sum_{i \in C} w_i > b$
- $\forall j \in C : \sum_{i \in C : i \neq j} w_i \leq b$
- $\sum_{i \in C} w_i + \max\{w_j : j \in N \backslash C\} - \max\{w_i : i \in C\} \leq b$

[T/F]: *The inequality $\sum_{i \in C} x_i \leq |C| - 1$ defines a facet of P_{KN}.*

(f) [6 pts] Let $\{f_0, f_1, ..., f_m\}$ be nonlinear functions. Consider the binary nonlinear optimization problem:

$$\boxed{\begin{aligned} & Z_{BP} = \min \sum_{j=1}^{n} f_0(x_j) \\ & \text{subject to} \\ & \sum_{i=1}^{n} f_j(x_i) \leq b_j, \ j = 1, \ldots, m \\ & \qquad x \in \{0,1\}^n \end{aligned}}$$

[T/F]: *The problem can be reformulated as a linear integer optimization problem.*

(g) [5 pts] Suppose we carry out the lift and project method in the variables $\{x_1, ..., x_k\}$.
[T/F]: *The order of the variables in which we perform the lift and project method may lead to different polyhedra.*

(h) [6 pts] Consider the robust optimization problem:

$$\boxed{\begin{array}{l} Z_R = \max c'x \\ \text{subject to} \\ \begin{array}{rcll} a'x & \leq & b & \forall a \in \{0,1\}^n : w'a \leq B \\ x & \geq & 0 & \end{array} \end{array}}$$

[T/F]: *Calculating Z_R is NP-hard.*

(i) [5 pts] Referring to (h), let $w = e$ (vector of 1's).
[T/F]: *Calculating Z_R is polynomially solvable.*

Solution:

(a) True. The absolute value function is piecewise linear.

(b) False. Consider a bi-partition matroid, a set T_1 containing one element in each partition and a set T_2 containing two elements in one partition. Then $|T_1| = |T_2| = 2$ but $r(T_1) = 2 \neq 1 = r(T_2)$.

(c) False. Since each integral x generated satisfies $Ax \leq b$ and x^* minimizes $c^T x$ over this set, we must have $P(c^T x < c^T x^*) = 0$. Since we have $E[c^T x] = c^T x^*$, we must then also have $P(c^T x > c^T x^*) = 0$. The result follows.

(d) False. $dim(P) \geq dim(F) = 3 \neq 0$ so $P \neq \emptyset$. We have $dim(P) = n - rank(A) = 4$, thus F is a facet and $fx \geq g$ therefore cannot be dropped.

(e) True. We apply theorem A.2 from the book. Let $fx \leq g$ represent the constraint $\sum_{i \in C} x_i \leq |C| - 1$ and F be the face it induces. Let $hx = d$ by any equality the holds for all $x \in F$. For any pair of elements $i, j \in C$ consider the solutions x^i and x^j which select all elements in $C \backslash \{i\}$ and $C \backslash \{j\}$ respectively and no elements from $N \backslash C$. We have $x^i, x^j \in F$ by the second property and thus $hx^i = hx^j \Rightarrow h_j = h_i$. Next pick $i = argmax\{w_i : i \in C\}$, for each element $k \in N \backslash C$ consider the solution $x^{i,k}$ which selects the elements in $\{k\} \cup C \backslash \{i\}$ and the solution $x^i \in F$ constructed as before. We have $x^{i,k} \in F$ by the third property and thus $hx^i = hx^{i,k} \Rightarrow 0 = h_k$. Thus $hx = \alpha f$ for some α.

(f) True. This can be accomplished by the following:

$$\boxed{\begin{array}{l} \min \sum_{i=1}^{n} y_{i,0} \\ \text{subject to} \\ \begin{array}{rcll} y_{i,j} & \geq & f_j(1) x_i & i = 1, ..., n, j = 0, ..., n \\ y_{i,j} & \geq & f_j(0)(1 - x_i) & i = 1, ...n, j = 0, ..., n \\ \sum_{i=1}^{n} y_{i,j} & \leq & b_j, \ j = 1, \ldots, m & \\ x & \in & \{0,1\}^n & \end{array} \end{array}}$$

or by replacing $f_j(x_i) = (f_j(1) - f_j(0)) x_i + f_j(0)$ in the original formulation.

(g) False. Any order provides the set $conv(P \bigcap_{i=1}^{k} \{x_k \in \{0,1\}\})$.

(h) True. By the equivalence of separation and optimization, the separation problem has the same complexity as calculating Z_R. But solving the separation problem is equivalent to solving KNAPSACK.

(i) True. The uncertainty set is now integral (since w is TU), so the separation problem is an LP and is thus polynomially solvable.

Problem (2: A directed cut formulation of MST-25 pts) Given a undirected graph $G = (V, E)$, with $|V| = n$ and $|E| = m$, form a directed graph $D = (V, A)$ by replacing each edge $\{i, j\}$ in E by arcs (i, j) and (j, i) in A. We select a node $r \in V$ as the root node. Let $y_{ij} = 1$ if the tree contains arc (i, j) when we root the tree at node r (in other words the solution will be a tree with directed edges away from the root). Let $\delta^+(S)$ be the set of arcs going out of S. Define:

$$\begin{aligned} P_{dcut} &= \big\{x \in \mathbb{R}^m : 0 \le x_e \le 1, x_e = y_{ij} + y_{ji}, \forall e \in E, \\ &\quad \sum_{e \in A} y_e = n-1, \sum_{e \in \delta^+(S)} y_e \ge 1, r \in S, \forall S \subset V, y_e \ge 0\big\} \\ P_{sub} &= \big\{x \in \mathbb{R}^m : 0 \le x_e \le 1, \forall e \in E, \sum_{e \in E} x_e = n-1 \\ &\quad \sum_{e \in E(S)} x_e \le |S| - 1, \forall S \subset V, S \ne \emptyset, V\big\} \end{aligned}$$

Prove $P_{dcut} = P_{sub}$.

Solution: For $x \in P_{sub}$ an extreme point, we construct a solution y by assigning positive flow emanating from the root node with value x_e; in other words we assign $y_{ij} = x_e$ with $e = (i,j)$ if (i,j) emanates from r. Such an assignment exists since P_{sub} is integral and has no cycles. For $r \in S$ we have $\sum_{e \in E(S)} x_e \le |S| - 1 \Rightarrow 1 \le \sum_{e \in \delta(S)} x_e = \sum_{e \in \delta^+(S)} y_e$. So such a solution $(x, y) \in P_{dcut}$ so $P_{sub} \subseteq P_{dcut}$.

For $x \in P_{dcut}$, $\exists y : \sum_{x \in E} x_e = \sum_{e \in A} y_e = n-1$ and $\sum_{e \in \delta^+(S)} y_e \ge 1$ for $S \ni r$. For each $i \in S \backslash \{r\}$ we have $\sum_{e \in \delta^-(\{i\})} y_e \ge 1$; since $\sum_{e \in A} y_e = n-1$ this then implies that $\sum_{e \in \delta^-(\{i\})} y_e = 1$ and $\sum_{e \in \delta^-(\{r\})} y_e = 0$.
For any $S | r \in S$ we then have $\sum_{x_e \in E(S)} x_e \le \sum_{v \in S} \sum_{e \in \delta^-(\{v\})} y_e \le |S| - 1$.
For any $S | r \notin S$ we have $\sum_{x_e \in E(S)} x_e \le \sum_{v \in S} \sum_{e \in \delta^-(\{v\})} y_e - \sum_{e \in \delta^+(V \backslash S)} y_e \le |S| - 1$. So $x \in P_{dcut}$ and thus $P_{sub} \supseteq P_{dcut}$.

Problem (3: Comparison of relaxations for the TSP-25 pts) Given an undirected graph $G = (V, E)$, consider the following two formulations of the TSP:

1)
$$\boxed{\begin{aligned} &\min \sum_{e \in E} c_e x_e \\ &\text{subject to} \\ &\sum_{e \in \delta(\{i\})} x_e = 2 \qquad \forall i \in V \\ &\sum_{e \in \delta(S)} x_e \ge 2 \qquad \forall S \subset V, S \ne \emptyset, V \\ &x_e \in \{0,1\} \quad \forall e \in E \end{aligned}}$$

2)
$$\boxed{\begin{aligned} &\min \sum_{e \in E} c_e x_e \\ &\text{subject to} \\ &\sum_{e \in \delta(\{i\})} x_e = 2 \qquad \forall i \in V \\ &\sum_{e \in E(S)} x_e \le |S| - 1 \quad \forall S \subset V, S \ne \emptyset, V \\ &x_e \in \{0,1\} \quad \forall e \in E \end{aligned}}$$

Let Z_{IP} be the common optimal cost of the two formulations. Let Z_1, Z_2 be the optimal cost of the linear relaxation of the two formulations respectively. Let Z_{D1}, Z_{D2} be the values of the Lagrangian duals if we relax the constraints $\sum_{e \in \delta(\{i\})} x_e = 2$ for all $i \ne 1$ in the two formulations. Let Z_{MST} be the cost of the minimum spanning tree with respect to the edge costs c_e. Order the values $Z_1, Z_2, Z_{IP}, Z_{D1}, Z_{D2}, Z_{MST}$.

Solution: We know that $Z_1 \le Z_{D1} \le Z_{IP}$ and $Z_2 \le Z_{D2} \le Z_{IP}$. We also know that the cut-set and subtour-elimination polyhedra are equivalent so $Z_1 = Z_2$. From example 4.5 in the book, we know that if we add the redundant (in the original problem) constraint $\sum_{e \in E(V \backslash \{1\})} = |V| - 2$ to the problem Z_{D2} to obtain a problem Z_D we have $Z_D = Z_2$, but without this extra constraint we have $Z_{D2} \le Z_D$, so $Z_{D2} = Z_2$. Examples such as the one in HW1 solutions show that the polyhedron for Z_{D1} is not necessarily integral. $Z_{MST} \le Z_2$ since any feasible point for Z_2 can be transformed into a feasible point for Z_{MST} with no greater cost. Thus we have:

$$Z_{MST} \le Z_1 = Z_2 = Z_{D2} \le Z_{D1} \le Z_{IP}$$