15.083: Integer Programming and Combinatorial Optimization
Problem Set 5 Solutions

Due 11/18/2009

Problem (1)

(a) We need only prove (iii) if and only if each non-empty set has a smallest element.
If (iii), then for any nonempty subset S, construct a sequence as follows: pick an arbitrary element of S and call it $\alpha(1)$, then keep adding elements from S to the sequence so that $\alpha(k+1) \prec \alpha(k)$ if there were not a smallest element in S we could repeat this process indefinitely arriving at a strictly decreasing sequence that does not terminate.
If each non-empty set S has a smallest element, then for any strictly decreasing sequence consider the set $S = \{\alpha(k)\}$. S must have a smallest element and thus the sequence must eventually terminate on this smallest element.

(b) Forall $\alpha, \beta \in \mathbb{Z}_+^n$ with $\alpha \neq \beta$, we have $\alpha - \beta \neq 0$ so the check for the leftmost entry being positive is well defined and lex is a total ordering. $\alpha - \beta = (\alpha + \gamma) - (\beta + \gamma)$ so property (ii) holds. Any vector $\alpha \in \mathbb{Z}_+^n$ can only be greater than $O(|\alpha|_\infty^b)$ other vectors with respect to the lex ordering. So any strictly decreasing sequence starting with $\alpha(1)$ can have at most $|\alpha(1)|_1$ terms before it terminates.

(c) Assume $\prec$ is a monomial ordering. For the sake of contradiction, assume $0 \succ \alpha$ for some $\alpha \in \mathbb{Z}_+^n$. Then the sequence $\alpha(k) = k\alpha$ is a strictly decreasing sequence that does not terminate.
Assume (i),(ii), and for all $\alpha \in \mathbb{Z}_+^n$, $\alpha \succeq 0$. By Dickson's Lemma, for any $A \subseteq \mathbb{Z}_+^n$ we have $\bigcup_{\alpha \in A} (\alpha + \mathbb{Z}_+^n) = \bigcup_{k=1}^{m} (\alpha(k) + \mathbb{Z}_+^n)$ for some m. We have $\beta \succeq 0$ for any $\beta \in \mathbb{Z}_+^n$. So the smallest element in the set $(\alpha + \mathbb{Z}_n^+)$ is α; since by Dickson's Lemma, for any $A \subseteq \mathbb{Z}_+^n$ we have $\bigcup_{\alpha \in A} (\alpha + \mathbb{Z}_+^n) = \bigcup_{k=1}^{m} (\alpha(k) + \mathbb{Z}_+^n)$ for some m. So the smallest element of A is one of the $\alpha(k)$'s; thus each set A has a smallest element.

(d) Let $f = \sum_{\alpha \in A} a_\alpha x^\alpha; g = \sum_{\beta \in B} b_\beta x^\beta$. Then $f \cdot g = \sum_{\alpha \in A} \sum_{\beta \in B} a_\alpha b \beta x^{\alpha + \beta}$. Let $\bar{\alpha} = \max_\succ \{\alpha \in A\} = multi(f)$ and $\bar{\beta} = \max_\succ \{\beta \in B\} = multi(g)$. Then by property (ii) $\bar{\alpha} + \bar{\beta} = \max_\succ \{\gamma \in A + B\} = multi(f + g)$.
$f + g$ introduces no new monomials that are not in f or g, though there may be cancelations. Suppose $multi(f + g) \succ \max\{multi(f), multi(g)\}$. Let x^γ be the monomial in $f + g$ that achieves $multi(f + g)$. Then x^γ is a monomial in f or g; but $multi(f), multi(g) \prec multi(f + g)$ arriving at contradiction.

Problem (2) Every monomial ideal is a Groebner basis. If division by the ideal results in zero remainder we obviously have ideal membership. Suppose we have ideal membership, but non-zero remainder. Then $I \ni f = \sum h_i(x) x^{\alpha(i)} + r \Rightarrow LT(r) \in < LT(x^{\alpha(i)}) >= I$ which means that r is divisible by at least one of the $\alpha(i)$ arriving at contradiction.

Problem (3)

(a)

(i) Since G is minimal $LT(g) = LT(g')$ which in turn makes G' minimal

(ii) g' is the remainder on division of $G \backslash \{g\}$ therefore by the division algorithm, no monomial of g' is divisible by $LT(G \backslash \{g\})$

(iii) An algorithm is simply repeating this process until all elements of G' are reduced.

(b) The reduced Groebner basis is $< x - 2y - 2w, z + 3w >$. Parametrically the family of solutions can be described for any s,t as : $(x, y, z, w) = (2s - 2t, s, -3t, t)$

Problem (4) The reduced Groebner basis obtained through Maple is $< 40y - 13y^2 + y^3, -8y + y^2 + 15x_6, 13y - y^2 - 40 + 40x_5, 5y - y^2 + 24x_4, 13y - y^2 - 40 + 40x_3, 5y - y^2 + 24x_2, x_1 - 1 >$. The minimum solution to the cubic equation for y is $y^* = 0$, propagating this value through the elimination ideal gives us an optimal solution of $(1, 0, 1, 0, 1, 0)$.

Problem (5)

(a) Checking all the vertices, we have an integral polytope: $\{(0, 0), (0, 3), (3, 0), (2, 2)\}$. The vectors $(1, 2)$ and $(2, 1)$ which induce the face $(2, 2)$ are not an integral generating set since they cannot generate $(1, 1)$, so we do not have a TDI system. By examining each vertex, we see that adding the constraints $x_1 + x_2 \leq 4$,$x_1 \leq 3$,$x_2 \leq 3$ gives us a TDI system.

(b) If we select t to be the least common integer multiple of the determinant of all submatrices B of A, we have for any c integral $A^T y = tc$ is integral by Cramer's rule. Thus we have such a TDI representation. Such a LCM exists by rationality.

(c)

(i) Given an integral solution, reverse the orientation of the arcs with $x_a = 1$. Then the number of arcs entering a proper subset are given by:

$$\sum_{a \in \delta^-(U)} (1 - x_a) + \sum_{a \in \delta^+(U)} x_a = |\delta^-(U)| - \sum_{a \in \delta^-(U)} x_a + \sum_{a \in \delta^+(U)} x_a \geq k$$

(ii) Set $x_a = \frac{1}{2}$ for all $a \in A$. Constraint (1) then reduces to $2k \leq |\delta^+(U)| + |\delta^-(U)|$ which is true by 2k-connectedness of G.

(iii) Consider maximizing $\sum_a c_a x_a$ over (1) and let z_U be the dual variable corresponding to the flow constraint for subset U. Amongst all dual optimal solutions, consider the one for which $\sum_{\emptyset \neq U \subset V} z^*_U |U| \cdot |V \backslash U|$ is minimal. Consider the set $\mathcal{F} = \{U : z^*_U > 0\}$. We claim that $\mathcal{F}$ is cross-free (ie satisfies $\forall U, T \in \mathcal{F} : U \subseteq T \textit{ or } U \supseteq T \textit{ or } U \cap T = \emptyset \textit{ or } U \cup T = V$. Suppose $U, T \in \mathcal{F}$ is a crossing. Then let $\epsilon = \min\{z_U, z_T\} > 0$. We can then define a new dual solution $\bar{z}$:

$$\begin{aligned} \bar{z}_U &:= z^*_U - \epsilon \\ \bar{z}_T &:= z^*_T - \epsilon \\ \bar{z}_{U \cap T} &:= z^*_{U \cap T} + \epsilon \\ \bar{z}_{U \cup T} &:= z^*_{U \cup T} + \epsilon \end{aligned}$$

and set all other components $\bar{z}_S := z^*_S$. $\bar{z}$ is still dual feasible and $(|\delta^-(U)| - k) + (|\delta^-(T)| - k) \geq (|\delta^-(U \cup T)| - k) + (|\delta^-(U \cap T)| - k)$ so $\bar{z}$ is dual optimal. Furthermore, $\sum_{\emptyset \neq U \subset V} \bar{z}_U |U| \cdot |V \backslash U| < \sum_{\emptyset \neq U \subset V} z^*_U |U| \cdot |V \backslash U|$ arriving at contradiction. Therefore the basis formed by the active dual constraints for the dual optimum, $\bar{z}$ correspond to a cross-free family $\mathcal{F}$ and in turn the basis matrix is totally unimodular. Thus for integral c, $\bar{z}$ is an optimal integer dual solution. Hence (1) is TDI.

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