

15.433 INVESTMENTS

**Class 7: The CAPM and APT**

**Part 2: Applications and Tests**

Spring 2003

# Predictions and Applications

- Predictions:
  - CAPM: In market equilibrium, investors are only rewarded for bearing the market risk.
  - APT: In the absence of arbitrage, investors are only rewarded for bearing the factor risk.
- Applications:
  - professional portfolio managers: evaluating security returns and fund performance.
  - regulatory commissions: cost of capital for regulated firms.
  - court rulings: evaluating claims of lost future income.
  - corporate manager: capital budgeting decisions.

# Testability of CAPM and APT

The wide acceptance of the CAPM and APT makes it all the more important to test their predictions empirically.

Recall from Class 6, both theories build on assumptions that are unrealistic at times.

How does a product of abstract reasoning hold in reality?

Unfortunately, the predictions of the CAPM and APT are hard to test empirically:

- Neither the market portfolio in CAPM nor the risk factor in APT is observable.
- Expected returns are unobservable, and could be time varying.
- Volatility is not directly observable, and is time varying.

# An Ideal Test of the CAPM

In an ideal situation, we have the following inputs:

1. Riskfree borrowing/ lending rate  $r_f$
2. Expected returns on the market  $E(r_M)$  and on the risky asset  $E(r_i)$ .
3. The exposure to market risk

$$\beta_i = \frac{\text{cov}(r_M, r_i)}{\text{var}(r_M)} \quad (1)$$

These inputs allow us to examine the relation between reward ( $E(r_i) - r_f$ ) and risk  $\beta_i$

1. More risk, more reward?
2. Do they line up?
3. What is the reward for a risk exposure of 1?
4. Zero risk, zero reward?

# A Linear Relation between Risk and Reward

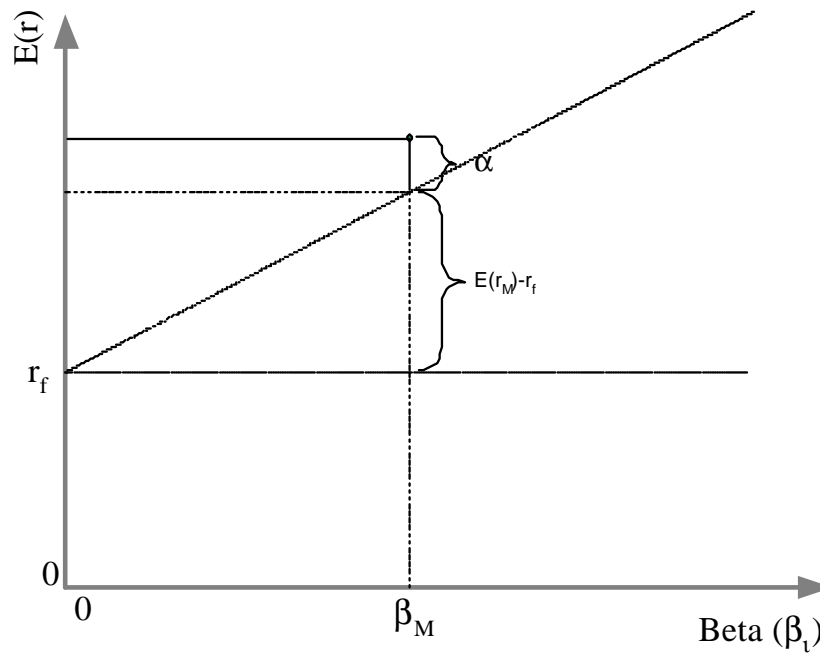


Figure 1: Beta with and without risk-free interest rate.

## Some Practical Compromises

the market portfolio  $r_M$  is unobservable: use a proxy, e.g., the S&P 500 index.

expected returns  $E(r_M)$  and  $E(r_f)$  are unobservable: use sample average

$$\mu_M = \frac{1}{N} \sum_t^T r_{M,t} \quad \mu_i = \frac{1}{N} \sum_t^T r_{i,t} \quad (2)$$

unobservable risk exposure

$$\beta_i = \frac{\text{cov}(r_M, r_i)}{\text{var}(r_i)} \quad (3)$$

use sample estimates:

$$\beta_i = \frac{\widehat{\text{cov}}(r_M, r_i)}{\widehat{\text{var}}(r_i)} \quad (4)$$

where

$$\widehat{\text{var}}(r_i) = \frac{1}{N} \sum_t^T (r_{t,M} - \mu_M)^2 \quad (5)$$

$$\widehat{\text{cov}}(r_M, r_i) = \frac{1}{N} \sum_t^T (r_{t,M} - \mu_M)(r_{t,i} - \mu_i) \quad (6)$$

# Testing the Linear Relation

pick a proxy for the market portfolio  $r_M$ , and record  $N$  monthly returns:

$$r_{t,M} : i = 1, \dots, N \quad (7)$$

for the same sample period, collect a sample of  $I$  firms, each with  $N$  monthly returns:

$$r_{t,i} : i = 1, \dots, I \quad \text{and} \quad t = 1, \dots, N \quad (8)$$

construct the sample mean for  $r_M$ .

for the  $i$ -th firm, construct the sample mean  $\mu_i$ , and the sample estimate  $\beta_i$ .

for  $i = 1, \dots, I$  test the linear relation:

$$\mu_i - r_f = \gamma_0 + \gamma_1 \beta_i \quad (9)$$

# Implications of the CAPM

$\gamma_0 = 0$ : zero exposure  $\rightarrow$  zero reward.

$\gamma_1 = \mu_M - r_f$ : one unit of exposure same reward as the market.

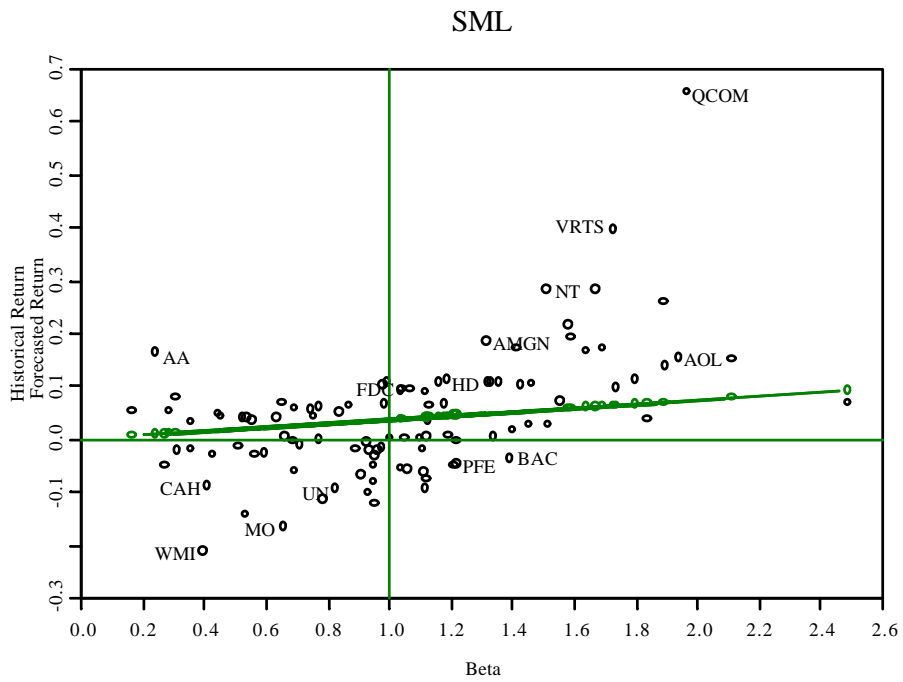


Figure 2: Betas for top 100 market weighted stocks from SP 500 against SP 500.



# Regression: The Basic Setup

Two variables  $x$  and  $y$ ,  $N$  pairs of outcomes:

$$(x_i, y_j), i = 1, 2, \dots, N. \quad (10)$$

We have reasons to believe that  $y$  and  $x$  are related. In particular, we would like to use  $x$  to explain  $y$ :

$$y_i = a + bx_j + \varepsilon. \quad (11)$$

$y$ : the dependent variable.

$x$ : the independent (explanatory) variable.

$\varepsilon_i$ : a random disturbance with zero mean.

coefficients: intercept  $a$ , slope  $b$ .

# Regression: Motivation

Some motivating examples:

1. On day  $i$ ,  $x_i$  is the temperature at Orlando,  $y_i$  is the price of the futures contract on frozen concentrated orange juice.
2. For firm  $i$ ,  $x_i$  is its leverage ratio,  $y_i$  is its probability of default.
3. On day  $i$ ,  $x_i$  is the Fed fund target rate,  $y_i$  is the 3 month T bill rate.
4. At the  $i$ -th second,  $x_i$  is the number of packets sent by *rgallati@mit.edu* to *jcox@mit.edu*,  $y_i$  is the number of packets received by the latter.

In each case, the outcome of  $x$  might contribute to the outcome of  $y$ :

$$y_i = a + b \cdot x_i + \varepsilon_i \tag{12}$$

but there might be other random factors, captured by  $\varepsilon_i$ , *that have nothing to do with  $x$* . The slope coefficient  $b$  is of particular interest as it measures the sensitivity of  $y$  to  $x$ .

# The Regression Coefficients

The objective: find  $a$  and  $b$  that best capture the linear relation between  $y$  and  $x$ . How: find  $a$  and  $b$  that minimize the squared differences:

$$\min_{a \in \mathbb{R}, b \in \mathbb{R}} \sum_{i=1}^N (y_i - a - b \cdot x_i)^2 \quad (13)$$

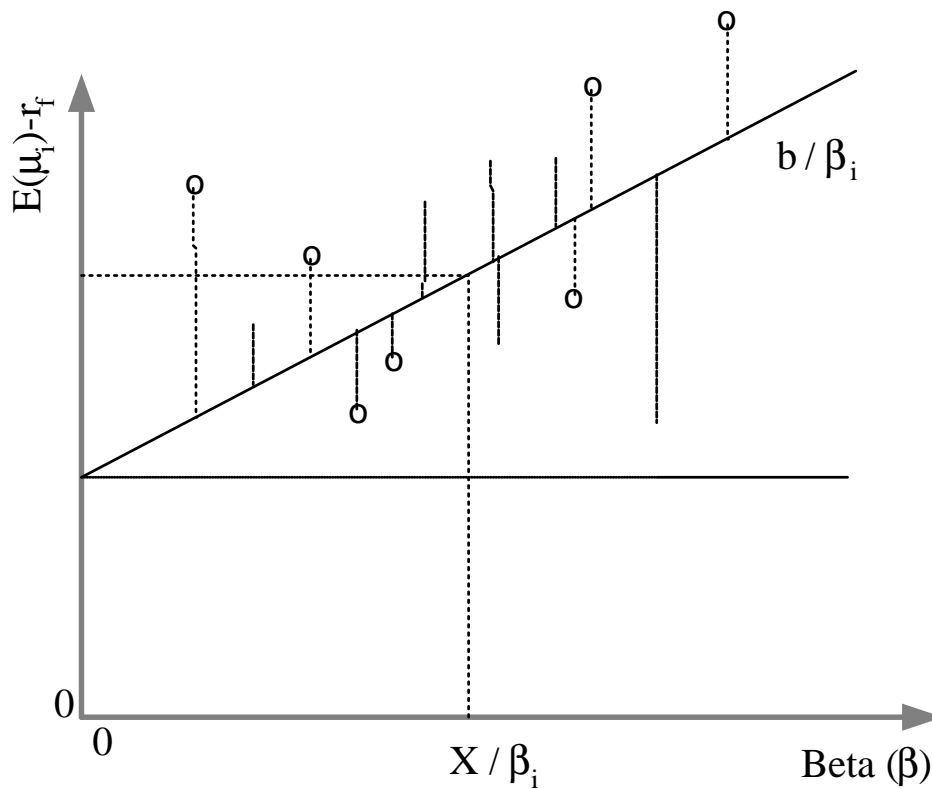


Figure 3: Regression.

# Regression: The Solution

Solving for the optimization problem, we get:

- an estimate  $\hat{b}$  for the slope coefficient  $b$ :

$$\hat{b} = \frac{\sum_{i=1}^N (y_i - \hat{\mu}_y) (x_i - \hat{\mu}_x)}{\sum_{i=1}^N (x_i - \hat{\mu}_x)} \quad (14)$$

- an estimate  $\hat{a}$  for the intercept  $a$ :  $\hat{a} = \mu_y - \hat{b}\mu_x$

Notice the familiar notation for sample means:

$$\hat{\mu}_x = \frac{1}{N} \sum_t x_i \quad \hat{\mu}_y = \frac{1}{N} \sum_t y_i \quad (15)$$

Why do we call our solutions estimates? Why put hats on  $b$  and  $a$ ?

Do we always get  $\hat{b}$  that is close to the real  $b$ ,  $\hat{a}$  hat is close to the real  $a$ ?

In the large sample (large  $N$ ), we are pretty confident that they do, why?

# The Standard Error

In Order to guess the real values of  $b$  and  $a$ , we use the data to help us.

Given  $N$  pairs of observations  $(y_i, x_i)$ , our regression solutions  $\hat{b}$  and  $\hat{a}$  are the best guesses. But we can never be 100% sure.

How to quantify our uncertainty about  $\hat{b}$  and  $\hat{a}$ .

We think of  $\hat{b}$  and  $\hat{a}$  as random variables. For any estimate, say  $\hat{b}$ , we can get an estimate of its standard deviation, which is usually referred to as the Standard Error. The standard error of an estimate is one measure of its precision.

# Interpreting the Regression Result

For the purpose of this class, you will use a "canned" regression package (e.g., Excel):

Input:  $(y_i, x_i)$ ,  $i = 1, \dots, N$  Output:

- the estimates  $\hat{b}$  and  $\hat{a}$  and
- their standard errors:  $s_{\hat{b}}$  and  $s_{\hat{a}}$
- their t statistics:  $t_{\hat{b}} = \hat{b}/s_{\hat{b}}$  and  $t_{\hat{a}} = \hat{a}/s_{\hat{a}}$
- the R squared

The standard errors and the t stat's provide measures of precision of your estimates.

The R squared tells you how much of the randomness in the dependent variable  $y$  is explained by the explanatory variable  $x$ .

# More on the Slope Estimate

Recall that our estimate for the slope coefficient  $b$ :

$$\hat{b} = \frac{\sum_{i=1}^N (y_i - \hat{\mu}_y) (x_i - \hat{\mu}_x)}{\sum_{i=1}^N (x_i - \hat{\mu}_x)} \quad (16)$$

Some familiar notation:

$$\widehat{var}(r_i) = \frac{1}{N} \sum_t^T (r_{t,M} - \mu_M)^2 \quad (17)$$

$$\widehat{cov}(r_M, r_i) = \frac{1}{N} \sum_t^T (r_{t,M} - \mu_M) (r_{t,i} - \mu_i) \quad (18)$$

We have

$$\beta_i = \frac{\widehat{cov}(r_M, r_i)}{\widehat{var}(r_i)} \quad (19)$$

Intuitively,  $b$  is a measure of the covariance  $\text{cov}(x,y)$  between  $x$  and  $y$ , scaled by the variance  $\text{var}(x)$  of  $x$ .

# Back to testing the CAPM

The implication of the CAPM:

$$\widehat{\mu}_i - r_f = \gamma_0 + \gamma_1 \beta_i \quad (20)$$

Our data  $(\widehat{\mu}, \widehat{\beta})$  on the 43 industrial portfolios tell us that this relation does not hold exactly.

One possibility: our measures of the expected returns are contaminated by noises that are unrelated to the  $\beta$ 's.

What we still like to know:

- On average, is reward related to risk at all?  $\gamma_1 = 0$  or not ?
- On average, does zero risk result in zero reward?  $\gamma_0 = 0$  or not?
- On average, does one unit of risk exposure pay the market return?

$$\gamma_1 = \mu_M - r_f = 5.9\% \text{ or not?} \quad (21)$$



# Regression in Action

Set up a regression:

- The dependent variable:  $y_i = \hat{\mu}_i - r_f$
- The independent variable:  $x_i = \hat{\beta}_i$
- Add noise  $\varepsilon_i$  that is unrelated to  $\beta_i$ .

Feed the data to the regression package:

	estimate	standard error	t-stat
$\gamma_0$	6%	1.8%	3.5
$\gamma_1$	0.17%	1.7%	0,1

R squared = 2%

Recall the implications of the CAPM:

1. The intercept  $\gamma_0 = 0$
2. The slope  $\gamma_1 = \hat{\mu}_M - r_f = r_f$

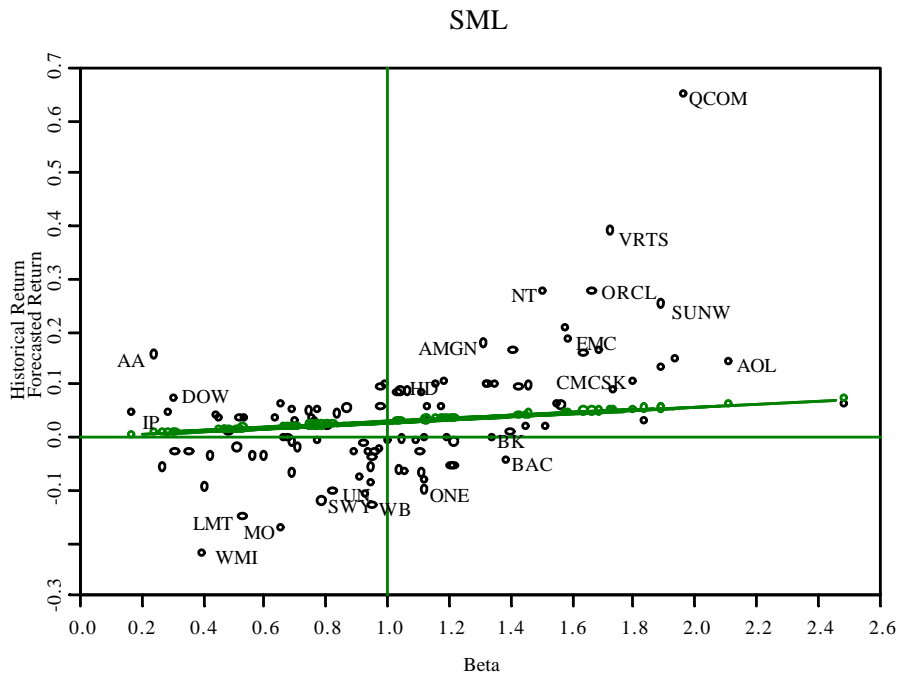


Figure 4: Betas (without  $r_f$ ) for top 100 market weighted stocks from S&P 500 against S&P 500.

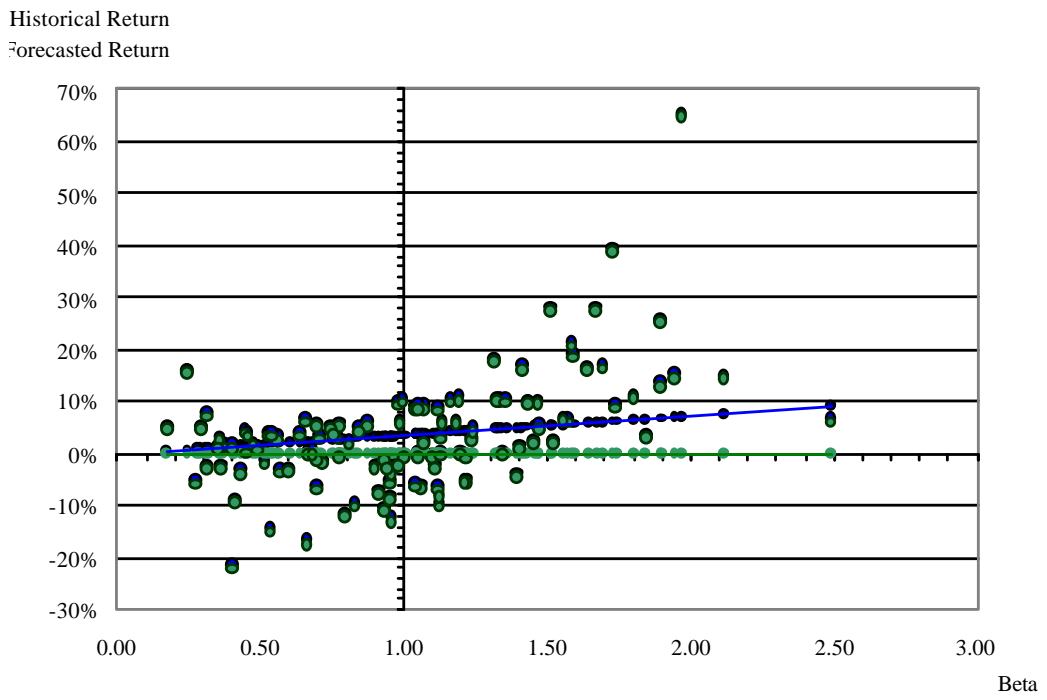


Figure 5: Betas (with and without  $r_f$ ) for top 100 market weighted stocks from S&P 500 against S&P 500.

## A Rule of Thumb with t-Stat

To get a sense of how an estimate, say  $\hat{b}$ , differs significantly from zero, its t-stat  $\hat{t}_b$  is the most telling statistics.

A Rule of Thumb: think of  $\hat{t}_b$  as standard normal (not a bad assumption for a large sample). The larger the magnitude (absolute value) of  $\hat{t}_b$ , the more likely it is significantly different from zero.

Hypothesis Testing: The null:  $\hat{b} = 0$ , the alternative:  $\hat{b} \neq 0$ .

1. A t-stat of 1.960 rejects the null with significance level 5%;
2. A t-stat of 2.576 rejects the null with significance level 1%;

For example,  $\hat{t}_{\hat{\gamma}_1} = 0.1$ , what can we say about  $\hat{\gamma}_1$ ? What about  $\hat{\gamma}_1$ ?

# A Summary of the CAPM Tests

In general, the test results depend on the sample data, sample periods, statistical approaches, proxy for the market portfolio, etc. But the following findings remain robust:

- The relation between risk and reward is much flatter than that predicted by the CAPM  $\hat{\gamma}_1 = \hat{\mu}_i - r_f$ .
- The risk measure  $\beta$  cannot even begin to explain the cross sectional variation in the expected returns. ( $\hat{\gamma}_1$  is statistically insignificant, R squared is close to zero.)
- Contrary to the prediction of the CAPM, the intercept  $\hat{\gamma}_0$  is significantly different from zero.

# Some Possible Explanations

1. Is the stock market index a good proxy for the market portfolio?
  - only 1/3 non governmental tangible assets are owned by the corporate sector.
  - among the corporate assets, only 1/3 is financed by equity
  - what about intangible assets, like human capital?
  - what about international markets?
2. Measurement error in  $\beta$ :
  - Except for the market portfolio, we never observe the true  $\beta$ .
  - To test the CAPM, we use estimates for  $\beta$ , which are measured with errors.
  - The measurement error in  $\beta$  will cause a downward biased estimate for the slope coefficient, and an upward biased estimate for the intercept.
3. Measurement error in expected returns
  - we use sample means  $\hat{\mu}_M$  and  $\hat{\mu}_i$  as proxies for the real, unobservable expected returns
  - it is known that means are hard to estimate, and there are noises in our estimates  $\hat{\mu}_M$  and  $\hat{\mu}_i$  and
  - if the noises in  $\hat{\mu}_M$  and  $\hat{\mu}_i$  are correlated, then we have a statistical problem (errors in variables)
4. Borrowing restrictions

- This class covers only one version of the CAPM, assuming unrestricted borrowing.
- In practice, borrowing restrictions are realistic. It includes margin rules, bankruptcy laws that limit lender access to a borrower's future income, etc.
- Fisher Black showed that borrowing restrictions might cause low- $\beta$  stocks to have higher expected returns than the CAPM predicts.

# Going Beyond the CAPM

Is  $\beta$  a good measure of risk exposure? What about the risk associated with negative skewness?

Could there be other risk factors?

Time varying volatility, time varying expected returns,

time varying risk aversion, and time varying  $\beta$ ?

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## Focus:

BKM Chapter 13

- p. 383 (13.1)
- p. 386 to 392 (beta, CAPM, SML, market index, concept check question 3 & 4),
- p. 391 to 393 top (13.2)
- p. 399 bottom (13.4 to 13.6)

Reader: Kritzman (1993) and Kritzman (1994).

type of potential questions: concept check question 1, 2, 3 & 4

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# Preparation for Next Class

Please read:

- Fama and French (1992) and
- Jegadeesh and Titman (1993).