# 15.433 INVESTMENTS <br> Classe 15: Forwards, Futures \& Swaps 

Spring 2003

## Interest Rate Derivatives

Interest rate swaps, caps, floors, and swaptions are over the counter (OTC) interest rate derivatives.

Broadly defined, a derivative instrument is a formal agreement between two parties specifying the exchange of cash payments based on changes in the price of a specified underlying item or differences in the returns of different securities.

For example, interest rate swaps are based on differences be-tween two different interest rates, while interest rate caps/floors are option like instruments on interest rates.

Unlike the organized exchanges, the OTC market is an informal market consisting of dealers or market makers, who trade price information and negotiate transactions over electronic communications networks.

Although a great deal of contract standardization exists in the OTC market, dealers active in this market custom tailor agreements to meet the specific needs of their customers.

And unlike the organized exchanges, where the exchange clearinghouses guarantee contract performance through a system of margin requirements combined with the daily settlement of gains or losses, counterparties to OTC derivative agreements must bear some default or credit risk.

According to data released by BIS, the total estimated notional amount of outstanding OTC contracts stood at \$94 trillion at end June 2000. The interest segment expanded by $7 \%$, to $\$ 64.1$ trillion.

- The total notional amount of interest rate swaps is $\$ 48$ trillion, among which, $\$ 17$ trillion goes to USD swaps.
- At the same time, the total public debt (including Treasuries and local government debt) outstanding in the U.S. is $\$ 5$ trillion.
- The total notional amount of interest rate options is 9 trillion.


## The History of Swaps

A swap contract is an obligation to pay a fixed amount and receive a floating amount at the end of every period for a pre-specified number of periods.

Each date on which payments are made is called a settlement date. If the fixed amount exceeds the floating amount on a settlement date, the buyer of the swap pays the seller the difference in cash. If the floating amount exceeds the fixed amount on a settlement date, the seller of the swap pays the buyer the difference in cash. In either case, this difference is conveyed in the form of a difference check.

A swap contract is equivalent to a portfolio of forward contracts with identical delivery prices and different maturities. Consequently, swap contracts are similar to forwards in that:

- at any date, swap contracts can have positive, negative, or no value.
- at initiation, the fixed amount paid is chosen so that the swap contract is costless. The unique fixed amount which zeros out the value of a swap contract is called the swap price.


Figure 1: Fixed/floating side of swap-contract

Our coverage of swaps will be limited as it is covered in depth in the Money and Capital Markets class. We will focus on the pricing and hedging of these instruments.

The first swap was a currency swap involving the World bank and IBM in 1980.

The first interest rate swap was a 1982 agreement in which the Student Loan Marketing Association (Sallie Mae) swapped the interest payments on an issue of intermediate term, fixed rate debt for floating rate payments indexed to the three month Treasury bill yield.

In a (standard) currency swap, the buyer receives the difference between the dollar value of one foreign currency unit (e.g., the dollar
value of 1 pound) and a fixed amount of domestic currency (e.g., 2 dollars) at every settlement date (e.g., every 6 months). Since the spot exchange rate at each settlement date is random, the dollar value of the foreign currency unit is also random. If this difference is negative, the swap buyer pays the absolute value of the difference to the swap seller.

Similarly, in an equity index swap, the buyer receives the difference between the dollar value of a stock index and a fixed amount of dollars at every settlement date.

The most common type of swap is an interest rate swap. The buyer of an interest rate swap receives the difference between the interest computed using a floating interest rate (e.g., LIBOR) and the interest computed using a fixed interest rate (e.g., $5 \%$ per 6 months) at every settlement date (e.g., every 6 months). The interest is computed by reference to a notional principal (e.g., $\$ 100$ mio.), which never changes hands.

If the floating side of an equity index or interest rate swap is calculated by reference to an index or interest rate in a foreign country, then there is currency risk, in addition to the usual equity or interest rate risk.

If the currency is negatively correlated with the underlying, this currency risk is actually desirable. Nonetheless, many swap contracts are quoted with a fixed currency component, so that only the equity or interest rate risk remains.

## The Growth of the Swap Market

Almost all swaps are traded over-the-counter (OTC). However, exchanges are now trying to list swaps.

Swaps usually go out from 2 years up to 10 years. This is called the tenor of the swap.

It was estimated by the International Swap Dealers Association (ISDA) that as of June 30, 1997 about US $\$ 23.7$ trillion worth of currency and interest rate swaps notional value existed, of which more than $93 \%$ comprise interest rate swaps. (Source: ISDA's Web Page at http://www.isda.org/)

Initially, banks acted as intermediaries trying to match a swap buyer with a seller. Over time, banks emerged as counterparties to swaps as they learned to lay off their risk using other derivatives such as exchange-traded futures. This is known as warehousing and the transactions are referred to as "matched-book" trading.

Using futures to hedge swap books is cost-effective but results in hedging complexities due to the non-linear relationship between the swap book, which is a portfolio of forward contracts (to be shown!), and futures contracts. Recall, there is a slight difference between forward and futures contracts. We defer this discussion to the end of this handout.

Practitioners use ad hoc as well as model-based methods to ad-just for the spread (a.k.a. convexity spread) which exists between forward contracts and corresponding futures contracts when hedging swap books.

## Example of an Interest-Rate Swap

This example is drawn from Hull, pages 147-149.

Assume A wants a floating rate loan. Usually the floating rate loan is pegged at LIBOR of a particular term (+ spread). The LIBOR term is set equal to the settlement date periodicity. For example, if settlement is every six months, the floating rate is six-month spot LIBOR (prevailing over the next six months).

Assume B wants a fixed rate loan. Usually the fixed rate loan is pegged at the yield-to-maturity of the Treasury security of the swap tenor (+ spread).

The (non-swap) borrowing structure for Parties A and B is as follows:
S\&P Credit Rating Party Fixed (\%) Floating (\%)

| AA | A | 10 | LIBOR +0.3 |
| :--- | :---: | :---: | :---: |
| BBB | B | 11.2 | LIBOR +1.0 |
| Gain: | $\Delta_{B-A}$ | 1.2 | 0.7 |

We say B has a comparative advantage in floating rate loans. Hence, it can "offer" a floating rate loan to A. Similarly, A can "offer" a fixed rate loan to B. These "offers", using an interest rate swap, beat the rates at which they can borrow by going to the market for loans directly.

Comparative advantage gain $=(1.2-0.7) \%=0.5 \%$ (50 b.p.). Note:

100 basis points (b.p.) $=1 \%$
Parties A and B can agree to split the gain $\Delta_{B-A}^{\text {floating }}-\Delta_{B-A}$ each party can improve its borrowing by $0.25 \%$ ( 25 b.p.).

Recall: B wants fixed, A wants floating. They can enter into a direct interest rate swap agreement as follows:


Figure 2: Example of interest-rate-swap.

Assuming the face values of Party A's and B's liabilities (i.e., their short positions in fixed rate and floating rate bonds, respectively) are $\$ 100 \mathrm{M}$ each, the notional principal value of the swap can be set at $\$ 100 \mathrm{M}$, too. This notional principal never changes hands.

Net Result:

- B borrows at:
$($ LIBOR $+1 \%)-$ LIBOR + fixed $=10.95 \%$
fixed leg of swap $=9.95 \%$
(Note: $10.95 \%$ was obtained by subtracting Party B's swap gain of $25 \mathrm{~b} . \mathrm{p}$. from its fixed rate (non-swap) loan of $11.2 \%$ )
- Therefore, A borrows at:

$$
10 \%+\text { LIBOR }-9.95 \%=\text { LIBOR }+0.05 \%
$$

The floating leg of the swap was picked to be LIBOR exactly. It could have been picked to be LIBOR + spread, we would have to increase the fixed leg of the swap by the spread amount, too, in order to achieve parity!

## A Synthetic Swap



Floating for Fixed Swap $=$ Fixed - Floating

Figure 3: Example of interest-rate-swap.

Suppose that the swap is of notional \$ 100, with semi-annual settlements. Let $r_{t}$ be the time- $t 6$-month LIBOR.

Let the swap rate be $s$, known at time 0 .
On the $i$-th settlement date $\tau_{i}$, an investor on the fixed side of the swap receives $\$ 100 \mathrm{~s} / 2$, and pays $\$ 100 r_{\tau_{i-1}} / 2$. (plain vanilla interest rate swap).

A synthetic swap is similar:
Floating-For Fixed Swap $=$ Fixed - Floating

The time-0 value of the fixed:

$$
\begin{equation*}
A_{0}=\sum_{i=1}^{8} \frac{\frac{\$ 100 \cdot s}{2}}{\left(\frac{1+r_{0, \tau_{i}}}{2}\right)^{i}}+\frac{\$ 100}{\left(\frac{1+r_{0,4}}{2}\right)^{8}} \tag{1}
\end{equation*}
$$

The time- 0 value of the floating: $B_{0}=\$ 100$

For a swap starting at time 0 , the swap rate s is set so that $A_{0}=B_{0}$.

## Swap Valuation

Suppose that the swap has a maturity of T years, with semi-annual settlements. Let

$$
\begin{equation*}
t=0.5,1,1.5, \ldots, T \tag{2}
\end{equation*}
$$

be the half year dates between 0 and $T$.

For notational simplicity, let $D(t, n)$ denote the time t value of a dollar payment to be made at time $\mathrm{t}+\mathrm{n}$ :

$$
\begin{equation*}
D(t, n)=\frac{1}{\frac{1+r_{t, n} 2 n}{2}} \tag{3}
\end{equation*}
$$

Using the synthetic argument, we can calculate the time 0 swap rate so as:

$$
\begin{equation*}
s_{0}=2 \cdot \frac{1-D(0, T)}{\sum_{i=1}^{2 \cdot T} D(0, i / 2)} \tag{4}
\end{equation*}
$$

We can see that the swap rate so is, in fact, the par rate of a coupon bearing bond. That is, so is the coupon rate of a bond sold at par.

## Swaptions

Swaptions are options on interest rate swaps.

An in- $\tau$ for T European swaption is a contract that gives the holder the right, not the obligation, to enter a T year swap at some future date at a pre known rate K . Let $s_{\tau}$ be the time- $\tau$ T-year swap rate. The payoff of the swaption at the exercise date $\tau$ :
$\left(K-s_{\tau}\right)^{+} P_{\tau} / 2$ (receiver's swaption)
$\left(s_{\tau}-K\right)^{+} P_{\tau} / 2$ (payer's swaption),
where
$P_{\tau}=D(\tau, 0.5)+D(\tau, 1)+, \ldots,+D(\tau, T)$ is the time value of the stream of dollars paid semi annually between $\tau$ and $\tau+\mathrm{T}$.

## American Swaptions



Figure 4: Swap cash flows, receive fixed.

## Swaptions Valuation

At the exercise date $\tau$, the moneyness of the swaption depends on the T year swap rate $s_{T}$, at time $\tau$, which is unknown at time 0.

The usual approach (the Black-Model) is to assume that the swap rate $s_{\tau}$, is log-normal. The pricing formula is very similar to that of Black and Scholes.

The value of the swaption depends on the distribution, in particular, the volatility of $s_{\tau}$.

If the future swap rate $s_{\tau}$, is known for certain at time 0 , then the swaption is worthless.

## Caps and Floors

An interest rate cap gives the buyer the right, not the obligation, to receive the difference in the interest cost (on some notional amount) any time a specified index of market interest rates rises above a stipulated "cap rate".

Caps evolved from interest rate guarantees that fixed a maximum level of interest payable on floating rate loans.

The advent of trading in over the counter interest rate caps dates back to 1985 , when banks began to strip such guarantees from floating rate notes to sell to the market.

The leveraged buyout boom of the 1980s spurred the evolution of the market for interest rate caps. Firms engaged in leveraged buyouts typically took on large quantities of short term debt, which made them vulnerable to financial distress in the event of a rise in interest rates. As a result, lenders began requiring such borrowers to buy interest rate caps to reduce the risk of financial distress.

Similar in structure to a cap, an interest rate floor gives the buyer the right to receive the difference in the interest cost any time a specified index of market interest rates falls below a stipulated "floor rate".

## Swaptions and Caps

Caps and swaptions are generally traded as separate products in the financial markets, and the models used to value caps are typically different from those used to value swaptions.

Furthermore, most Wall Street firms use a piecemeal approach to calibrating their models for caps and swaptions, making it difficult to evaluate whether these derivatives are fairly priced relative to each other.

Financial theory, however, implies no arbitrage relations for the cap and swaption prices.

A cap can be represented as a portfolio of options on individual forward rates. In contrast, a swaption can be viewed as an option on the forward swap rate, which is a portfolio of individual forward rates.

This implies that the relation between cap and swaption prices is driven primarily by the correlation structure of forward rates.

## Spot and forward rates

With annual coupon payments, the Yield to Maturity (YTM) is defined by:

$$
\begin{align*}
\text { Bond Price } & =\sum_{t=1}^{N} \frac{C P N_{t}}{(1+Y T M)^{t}}+\frac{P A R}{(1+Y T M)^{N}}  \tag{5}\\
& =\text { Present Value of future payments }(Y T M)
\end{align*}
$$

Shortcomings of YTM valuation when the yield curve is not flat:

- all cash flows are discounted at a single rate;
- all cash flows are assumed to be reinvested at the YTM.

Spot rates: All cash flows occuring at the same time must have the same required yield (given that they have the same issuer):

$$
\begin{align*}
\text { Bond Price } & =\sum_{t=1}^{N} \frac{C P N_{t}}{(1+S)^{t}}+\frac{P A R}{(1+S)^{N}}  \tag{6}\\
& =\text { Present Value of future payments }\left(S_{t}\right)
\end{align*}
$$

Forward rates: The future short rates, at which you can place your money today (by buying a forward contract on the appropriate bond).

If we again only consider yearly payments and $S_{t}$ is the spot rate for year t , and where ${ }_{n} f_{m}$ is $n$-year the forward rate between year $m$ and year $m+n$, then we have:

$$
\begin{equation*}
\left(1+S_{2}\right)^{2}=\left(1+S_{1}\right) \cdot\left(1+{ }_{1} f_{1}\right) \Rightarrow{ }_{1} f_{1}=\frac{\left(1+S_{2}\right)^{2}}{\left(1+S_{1}\right)}-1 \tag{7}
\end{equation*}
$$

Argument: We should expect to have the same return whether we now buy the 2 year bond or now buy a bond that matures in a year and use the money received from this bond to pay the forward price for a bond
that at this time is a one year bond.

The forward rates can be given two interpretations; the break-even rate or the locked in - even though the two are very similar. The forward rates are break even rates in the sense that if they are realize all bond positions earn the same return. Therefore, if you for example expect rates to rise by less than what is implied by the forward rates you should take a bullish stance (buy bonds) as lower interest rates will imply higher bond prices.

Determining the (implied) forward rates from spot rates:
The general formula (annualized rate):

- Note that ${ }_{1} f_{0}=S_{1}$ and that if spot rates are increasing linearly, the forward rates will be increasing with a slope twice the slope of the spot rates,
- if the spot rates changes from increasing to decreasing, i.e. the yield curve is humped shaped, the forward rate curve will go through the maximum of the spot rates curve,
- if the spot rates are constant, so is the forward rates at the same level.

If we for example have $S_{1}=4 \%$ and $S_{2}=8.167 \% \Rightarrow{ }_{1} f_{1}=12.501 \%$.

Similarly : $(1.12377)^{3}=(1.04) \cdot(1.12501) \cdot\left(1+{ }_{1} f_{2}\right)$ or $(1.08167)^{2} \cdot\left(1+{ }_{1} f_{2}\right)$ both implies that ${ }_{1} f_{2}=21.295 \%$.

Determining spot rates from forward rates:

$$
\begin{equation*}
\left(1+S_{2}\right)^{3}=\left(1+{ }_{1} f_{0}\right)\left(1+{ }_{1} f_{1}\right)\left(1+{ }_{1} f_{2}\right) \tag{8}
\end{equation*}
$$

The general formula:

$$
\begin{equation*}
S_{t}=\left(\left(1+{ }_{1} f_{0}\right) \cdot\left(1+{ }_{1} f_{1}\right) \cdots\left(1+{ }_{1} f_{t-1}\right)\right)^{1 / t}-1 \tag{9}
\end{equation*}
$$

## Construction of the Spot Rate Curve

Several different sets of Treasures can be used to construct a default-free theoretical spot rate curve:

- On-the-run Treasury: The newest Treasury issues. Bootstrapping used to get the different yields.
- On-the-run Treasury issues and selected off-the-run Treasury issues: helps to avoid a large maturity gap in the on-the-run Treasuries. Note there is a tax effect for Treasuries not selling at par-the par coupon curve uses adjusted yields and linear interpolation to obtain the yield curve. Bootstrapping used to get the different yields.
- All treasury coupon securities and bills: All are used in the hope that they contain additional information relative to the above. here it is not possible to use bootstrapping as there may exist several yields for each maturity (uses instead exponential spline fitting).
- Treasury coupon strips: are observable zero/coupon securities but are normally not used as they include a liquidity premium and are associated with tax aspects even more complicated than those mentioned above.

Compute spot rates from the par yield curve (bootstrapping):

1. Get the prices on some usable bonds either directly or by using their YTM.
2. Use $\sum_{t=1}^{N} \frac{C P N_{t}}{\left(1+S_{t}\right)^{t}}+\frac{P A R}{\left(1+S_{N}\right)^{N}}$ to get the spot rate, $S_{t}$ one by one, first solve for $S_{1}$ and then use $S_{1}$ to solve for $S_{2}$ and so on.
3. Often the book argues that in step 3 we have to strip the coupons from the bond and value them as stand-alone instruments - but I find it easier to do everything as described in step 2.

The general formula for $S_{t}$ using the bond with maturity at time t is:

$$
\begin{equation*}
S_{t}=\frac{\text { payment at time } t}{\text { price }-P V \text { of all payments before time } t} \cdot \frac{1}{t}-1 \tag{10}
\end{equation*}
$$

Example: The information in the following table is given to you, reflecting the current market conditions.
Maturity YTM Coupon Price

| 1 yr. | $4 \%$ | $0 \%$ | $96.154 \%$ |
| :--- | :---: | :---: | :---: |
| 2 yr. | $8 \%$ | $8 \%$ | $100.00 \%$ |
| 3 yr. | $12 \%$ | $6 \%$ | $85.589 \%$ |

Calculate:

- the spot interest rate for $S_{1}$, and
- the spot interest rate for $S_{2}$, and
- the spot interest rate for $S_{3}$.

Solution: First we of course get that $S_{1}=4 \%$ and therefore $S_{2}$ is the solution to:

$$
\begin{align*}
100 & =\frac{8}{1.04^{1}}+\frac{8}{\left(1+S_{2}\right)^{2}}  \tag{11}\\
\frac{108}{\left(1+S_{2}\right)^{2}} & =100-\frac{8}{1.04^{1}}  \tag{12}\\
\left(1+S_{2}\right)^{2} & =\frac{108}{100-7.6923}  \tag{13}\\
S_{2} & =\sqrt{\frac{108}{92.3077}}-1=8.167 \% \tag{14}
\end{align*}
$$

And similarly, $S_{3}$ is then the solution to $85.589=\frac{6}{1.04^{1}}+\frac{6}{1.08167^{2}}+$ $\frac{6}{\left(1+S_{3}\right)^{3}} \Rightarrow S_{3}=12.3777 \%$

## Forwards

The value of a forward contract at the time it is first entered into is zero. At a later stage it may prove to have a positive or negative value. Suppose that f is the value today of a long forward contract that has a delivery price of K and hat $F_{0}$ is the current forward price for the contract. A general result, applicable to a forward contract on either an investment or consumption asset, is:

$$
\begin{equation*}
f=\left(F_{0}-K\right) e^{-r \cdot T} \tag{15}
\end{equation*}
$$

where, as usual, $T$ is the time to maturity of the contract and $r$ is the risk-free rate for a maturity $T$. To see why equation ?? is correct, we compare a long forward contract has a delivery price of $K$. The difference between the two is only in the amount that will be paid for the underlying asset at time $T$. Under the fist contract this amount is $F_{0}$; under the second contract it is $K$. A cash flow difference of $F_{0}-K$ at time $T$ translates to a difference of $\left(F_{0}-K\right) e^{-r \cdot T}$ today. The contract with a delivery price $F_{0}$ is, therefore, less valuable than the contract with delivery price $K$ by an amount $\left(F_{0}-K\right) \cdot e^{-r \cdot T}$. The value of the contract that has a delivery price of $F_{0}$ is by definition zero. It follows that the value of the contract with a delivery price of K is $\left(F_{0}-K\right) \cdot e^{-r \cdot T}$. This proves equation ??. Similarly, the value of a short forward contract with delivery price $K$ is:

$$
\begin{equation*}
\left(K-F_{0}\right) \cdot e^{-r \cdot T} \tag{16}
\end{equation*}
$$

Equation ?? shows that we can value a long forward contract on an asset by assuming that the price of the asset at the maturity of the forward contract is the forward price, $F_{0}$. This is because, when we can make
this assumption, a long forward contract will provide a payoff at time $T$ of $F_{0}-K$. This is worth $\left(F_{0}-K\right) \cdot e^{-r \cdot T}$, the value of the forward contract, today. Similarly, we can value a short forward contract on the asset by assuming that the current forward price of the asset is realized.

We can generalize the valuation of a forward contract with no income as following:

$$
\begin{equation*}
F_{0}=S_{0} \cdot e^{r \cdot T} \tag{17}
\end{equation*}
$$

We can generalize to argue that, when an investment asset will provide income with a present value of I during the life of a forward contract:

$$
\begin{equation*}
F_{0}=\left(S_{0}-I\right) \cdot e^{r \cdot T} \tag{18}
\end{equation*}
$$

The forward price for an investment asset providing a continuous dividend yield at rate q is:

$$
\begin{equation*}
F_{0}=S_{0} \cdot e^{(r-q) \cdot T} \tag{19}
\end{equation*}
$$

Using equation ?? in conjunction with equation ?? gives the following expression for the value of a long forward contract on an investment asset that provides no income.

$$
\begin{equation*}
f=S_{0}-K \cdot e^{-r \cdot T} \tag{20}
\end{equation*}
$$

Similarly, using equation ?? in conjunction with equation ?? gives the following expression for the value of a long forward contract on an investment asset that provides a known income with present value I.

Finally, using equation ?? in conjunction with equation ?? gives the following expression for the value of a long forward contact on an investment asset that provides a known dividend yield at rate q.

$$
\begin{equation*}
f=S_{0} \cdot e^{q \cdot T}-K \cdot e^{-r \cdot T} \tag{21}
\end{equation*}
$$

Example: Consider a six-month long forward contract on a non-dividend-paying stock. The risk-free rate of interest (with continuous compounding) is $10 \%$ per annum, the stock price is $\$ 25$, and the delivery price is $\$ 24$. In the case $S_{0}=25, r=0.10, T=0.5$, and $K=24$. From equation ??, the forward price, $F_{0}$, is given by:

$$
\begin{equation*}
F_{0}=25 \cdot e^{0.1 \cdot 0.5}=\$ 26.28 \tag{22}
\end{equation*}
$$

From equation ??, the value of the forward contract is:

$$
\begin{equation*}
f=(26.28-24) \cdot e^{-0.1 \cdot 0.5}=\$ 2.17 \tag{23}
\end{equation*}
$$

Alternatively, equation ?? yield:

$$
\begin{equation*}
f=25-24 \cdot e^{-0.1 \cdot 0.5}=\$ 2.17 \tag{24}
\end{equation*}
$$

## Foreign Currencies

The underlying asset in such contracts is a certain number of units of the foreign currency. We will define the variable $S_{0}$ as the current spot price, measured in dollars, of one unit of the foreign currency and $F_{0}$ as the forward price, measured in dollars, of one unit of the foreign currency. This is consistent with the way we have defined $S_{0}$ and $F_{0}$ for other assets underlying forward contracts. However, it does not always correspond to the way spot and forward exchanges rates are quoted. For all major exchange rates, except the British pound, a spot or forward exchange rate is normally quoted as the number of units of the currency that are equivalent to one dollar. For the British pound, it is quoted as the number of dollars per unit of the foreign currency.

A foreign currency has the property that the holder of the currency can earn interest at the risk-free interest rate prevailing in the foreign country. For example, the holder can invest the currency in a foreigndenominated bond. We define $r_{f}$ as the value of this foreign risk-free interest rate for a maturity T with continuous compounding. As before, $r$ is the domestic risk-free rate for this maturity.

The relationship between $F_{0}$ and $S_{0}$ is:

$$
\begin{equation*}
F_{0}=S_{0} \cdot e^{\left(r-r_{f}\right) \cdot T} \tag{25}
\end{equation*}
$$

This is the well-known interest rate parity relationship from international finance. To understand the relationship, suppose first that $F_{0}>S_{0}$. $e^{\left(r-r_{f}\right) \cdot T}$. An investor can:

- Borrow $S_{0} \cdot e^{-r_{f} \cdot T}$ in the domestic currency at rate $r$ for time $T$.
- Use the cash to buy spot $e^{-r_{f} \cdot T}$ of the foreign currency and invest
this at the foreign risk-free rate.
- Short a forward contract on one unit of the foreign currency.

The holding in the foreign currency grows to unit at time $T$ because of the interest earned. Under the terms of the forward contract, the holding is exchanged for $F_{0}$ at time $T$. An amount $S_{0} \cdot e^{r-r_{f} \cdot T}$ is required to repay the borrowing. A net profit of $F_{0}-S_{0} \cdot e^{r-r_{f} \cdot T}$, is therefore, made at time $T$.

Suppose next that $F_{0}<S_{0} \cdot e^{r-r_{f} \cdot T}$. An investor can:

- Borrow $e^{-r_{f} \cdot T}$ in the foreign currency at rate $r$ for time $T$.
- Use the cash to buy $S_{0} \cdot e^{-r_{f} \cdot T}$ of the domestic currency and invest this at the domestic risk-free rate.
- Take a long position in a forward contract on one unit of the foreign currency.

In this case, the holding in the domestic currency grows to $S_{0} \cdot e^{r-r_{f} \cdot T}$ at time $T$ because of the interest earned. At time $T$ the investor pays $F_{0}$ and receives one unit of the foreign currency. The latter is used to repay the borrowings. A net profit of $S_{0} \cdot e^{r-r_{f} \cdot T}-F_{0}$, is therefore, made at time $T$.

Note that equation is identical to equation (??) with $q$ replaced by $r_{f}$. This is not a coincidence. A foreign currency can be regarded as in investment asset paying a known dividend yield. The "dividend yield" is the risk-free rate of interest in the foreign currency. To see why this is so, note that interest earned on a foreign currency holding is denominated in the foreign currency. Its value when measured in the domestic currency is therefore proportional to the value of the foreign currency.

The value of a forward foreign exchange contract is given by equation ?? with $q$ replaced by $r_{f}$. It is:

$$
\begin{equation*}
f=S_{0} \cdot e^{-r_{f} \cdot T} \tag{26}
\end{equation*}
$$

Example: Suppose that the six-month interest rates in the United States and Japan are $5 \%$ and $1 \%$ per annum, respectively. The current yen/dollar exchange rate is quote as 100 . This means that there are 100 yen per dollar or 0.01 dollars per yen. For a six-month forward contract on the yen $S_{0}=0.01, r=0.05, r_{f}=0.01, T=0.5$. From equation (??), the forward foreign exchange rate as:

$$
\begin{equation*}
0.01 e^{(0.05-0.01) \cdot 0.5}=0.01020 \tag{27}
\end{equation*}
$$

This would be quoted as $1 / 0.010202$ or 98.02 .

## Focus:

BKM Chapter 16

- p. 485-498 (duration, convexity, eq. 16.2, 16.3, 16.4)
- p. 500-508 (immunization)
- p. 509-512
- p. 514-515 (swaps)

Style of potential questions: Concept check questions, p. 519 ff. question $1,3,4,10,26$

BKM Chapter 22

- p. 744-749
- p. 758-759

Style of potential questions: Concept check questions, p. 762 ff., question 4,13

BKM Chapter 23

- p. 767-772 (eq. 23.1)
- p. 786
- p. 790-794

Style of potential questions: Concept check questions,p. 797 ff . question $1,4,7,11,13,25$

## Preparation for Next Class

Please read:

- BKM chapter 14, and
- Duffie and Gârleanu (2001).

