Graphical Integration Exercises

Part Two: Ramp Functions

Prepared for the
MIT System Dynamics in Education Project
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1. Abstract

This paper is the second in a series of papers on developing skills in graphical integration. It will study the use of graphical integration to estimate the behavior of simple systems with linearly increasing or decreasing flows. The paper assumes you can estimate the behavior of models with constant flows. You should know how to graphically integrate constant flows as well as step functions. Also, you should be familiar with the terms "slope" and "area," and you should know how to calculate them.[1]

2. Introduction

Let us consider a simple system, that of a bathtub with a constant inflow through a faucet. We know that if we turn on the faucet to fill a tub, go outside for five minutes, and come back to find the tub one-fourth full, then it will take five more minutes to fill up halfway. It will then take five more minutes to be three-fourths full, and a total of twenty minutes for the tub to be completely full of water.

What we just did, essentially, is graphical integration without explicitly drawing graphs. We were able to perform this integration because the system in question was very simple. However, many systems are not as simple as the bathtub system. Thus it becomes difficult to predict how they will behave. Practice in using graphical integration will make it easier to use it to better estimate the behavior of many systems.

Although we have access to sophisticated computer programs that simulate models of great complexity, it is important that we can intuitively estimate and understand what we see on the graph pad before and after running a simulation. Graphical integration helps improve our understanding of the relation between structure and behavior and also to catch errors in the model built. In this second paper of the series, we will explore flows that take on linearly increasing or decreasing values. This linearly increasing or decreasing behavior is often called a "ramp function."

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[1] If you need to develop these skills, please refer to: Alice Oh, 1995. *Graphical Integration Exercises Part One: Exogenous Rates* (D-4547), System Dynamics in Education Project, System Dynamics Group, Sloan School of Management, Massachusetts Institute of Technology, Dec. 1, 19pp.

3. Flows as Ramp Functions

The first paper in the series, *Graphical Integration Exercises Part One: Exogenous Rates*, studied graphical integration of constant flows and step functions. We will now turn to explore flows as ramp functions. A flow ramp function is a flow that is increasing or decreasing linearly, and thus it is not constant over time. An example of a linearly increasing flow ramp function is the filling of a bathtub by constantly turning the faucet in the "on" direction. As time goes on, not only does the amount of water in the tub increase, but so does the rate at which water enters the tub from the faucet. Another example of a ramping flow function, only linearly decreasing, is filling the same bathtub with the faucet initially at its highest flow value, and steadily turning the valve to decrease the rate at which water enters the tub. The tub fills, more and more slowly, until the valve is turned all the way and the water ceases to flow into the tub.

For this paper we will not concern ourselves with why the rate increases or decreases linearly. We will simply study how we can estimate the effect of the flow on the behavior of the stock.

3.1 Linearly Increasing Flows Cause Parabolic Growth

In the first paper of the Graphical Integration Exercises series, we noted that *the slope of the stock is equal to the value of the flow.* The same principle applies to all flows irrespective of whether they are constant or variable. Let us first look at an example: an initially empty bathtub with a linearly increasing (also called positively ramping) net flow.

Figure 1 shows the behavior of such a flow.

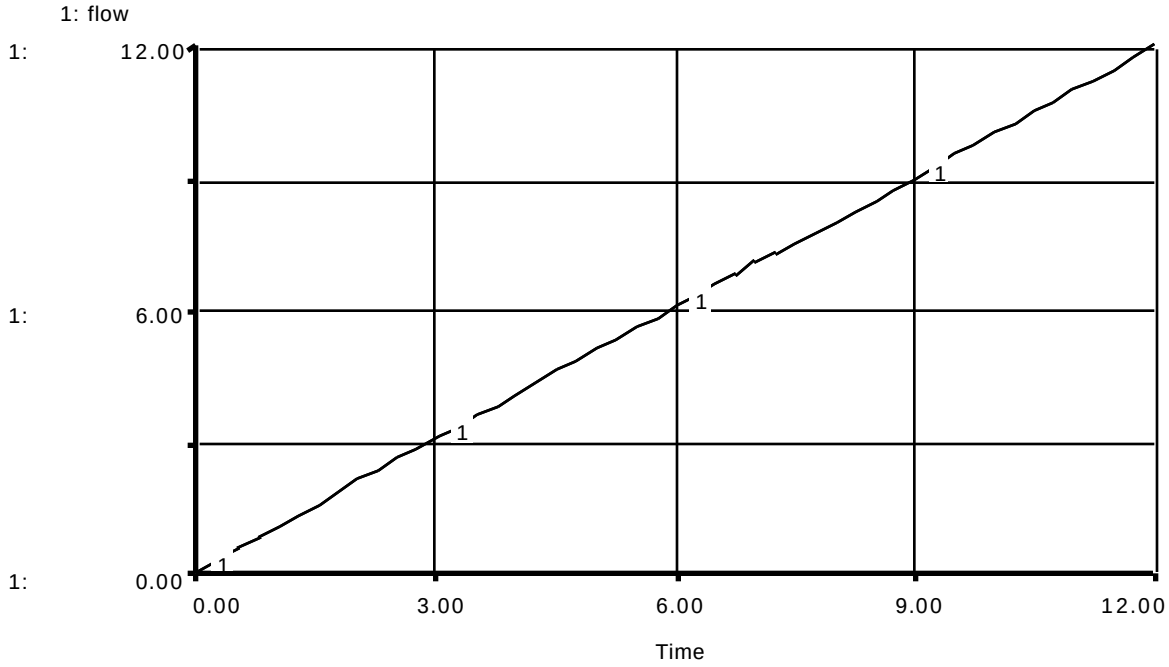

Figure 1: Linearly increasing flow with positive slope.

Figure 2 shows the resulting behavior of the stock.

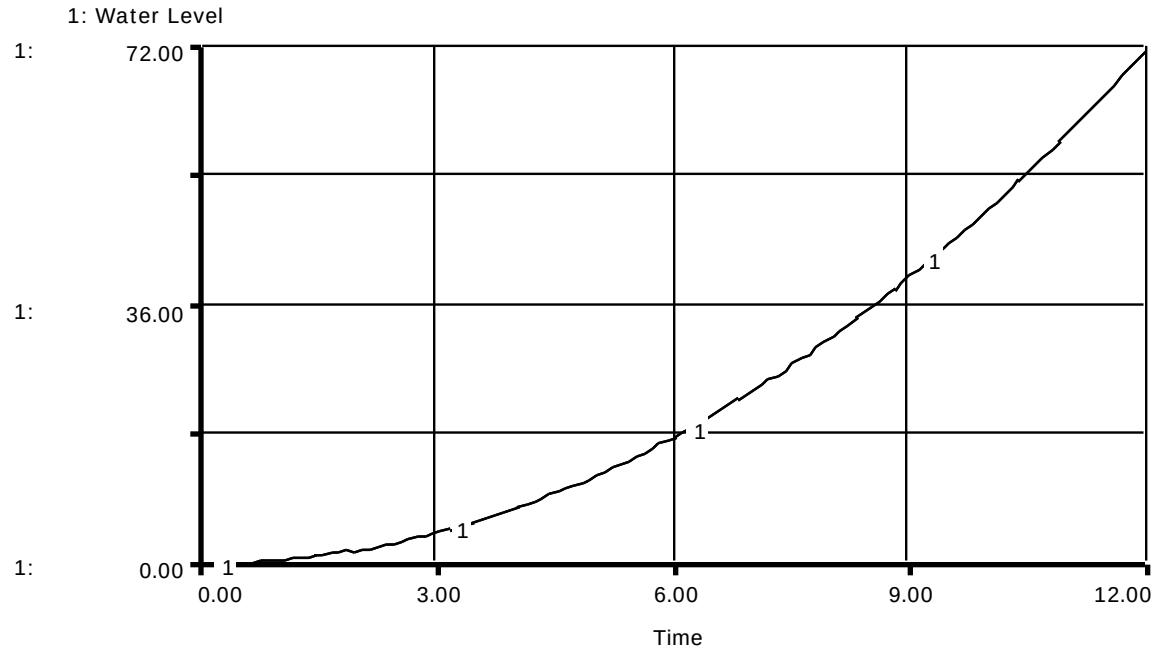

Figure 2: Behavior of stock with a linearly increasing flow.

Linearly increasing flows result in parabolic[2] growth of the stock. The area between the flow graph and the line representing a flow of zero equals the change in the stock over a time period. This is because the area, which is also the product of the flow value and time, is the amount added to the stock. We will explore this concept in greater detail a little later in the paper. You will find in graphical integration that calculating the area is extremely important for determining the behavior of the stock.

The area between the flow graph and the line representing a flow of zero in Figure 1 is the area of a triangle with a length of 12 units of time and a height of 12 units. Thus, the area is:

Total Area = 0.5 * base * height[3]

Total Area = 0.5 * 12 * 12

Total Area = 72

The reason that the change in the value of the stock equals the area between the flow curve and the line representing a flow of zero is that by calculating the area, what you are essentially doing is multiplying the flow by the time. This is the amount added to the stock. Thus, units per time times time are equal to the total number of units. Calculating the area under the flow graph is important if you are to integrate it graphically, because it allows you to easily determine the final value of the stock:

Final Value of Stock = Initial Value of Stock + Total Area

Final Value of Stock = 0 + 72

Final Value of Stock = 72

The parabolic behavior of the stock agrees with the fact that *the slope of the stock at any given time is equal to the value of the flow at that time.* We can calculate the slope of the stock at any point on the above graph by drawing the tangent line and calculating its slope. The tangent line at any given time is the line that touches the curve only at that point in time. Let us look at an example of a tangent line drawn into the curve shown in Figure 2.

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[2] For those of you mathematically inclined, parabolic growth means that the dependent variable is a quadratic equation: $y = ax^2 + bx + c$. Exponential growth is $y = ae^{bx}$.

[3] Remember, the area of a triangle is 0.5 times base times height.

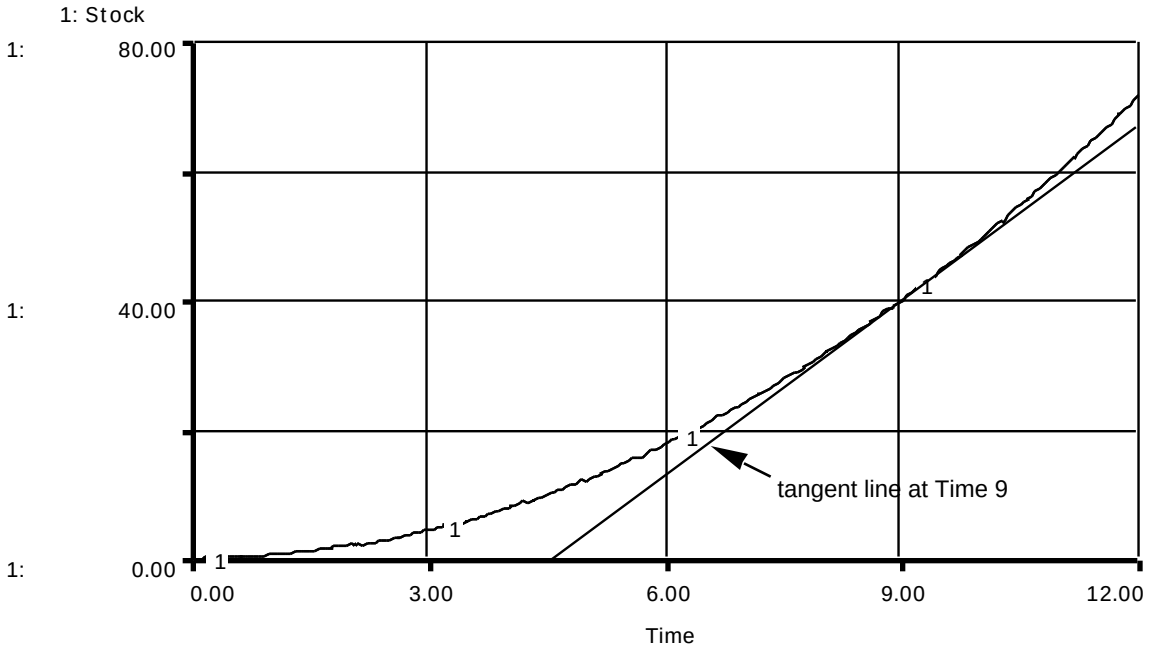


Figure 3: The behavior of an initially empty stock being filled with a linearly increasing flow.

Figure 3 shows the graph of the stock with the tangent line at Time 9. Notice how the slope of the tangent line is approximately (68-0)/(12-4.5) = 9.[4] Looking at Figure 1, we see that the value of the flow at Time 9 is also 9!

Now, let us look at another example. Let us consider again a linearly increasing flow, but with its start delayed until Time 10. We will assume the stock initially to be zero. Figure 4 shows the behavior of this flow.

[4] Slope = difference in y-coordinate/difference in x-coordinate.

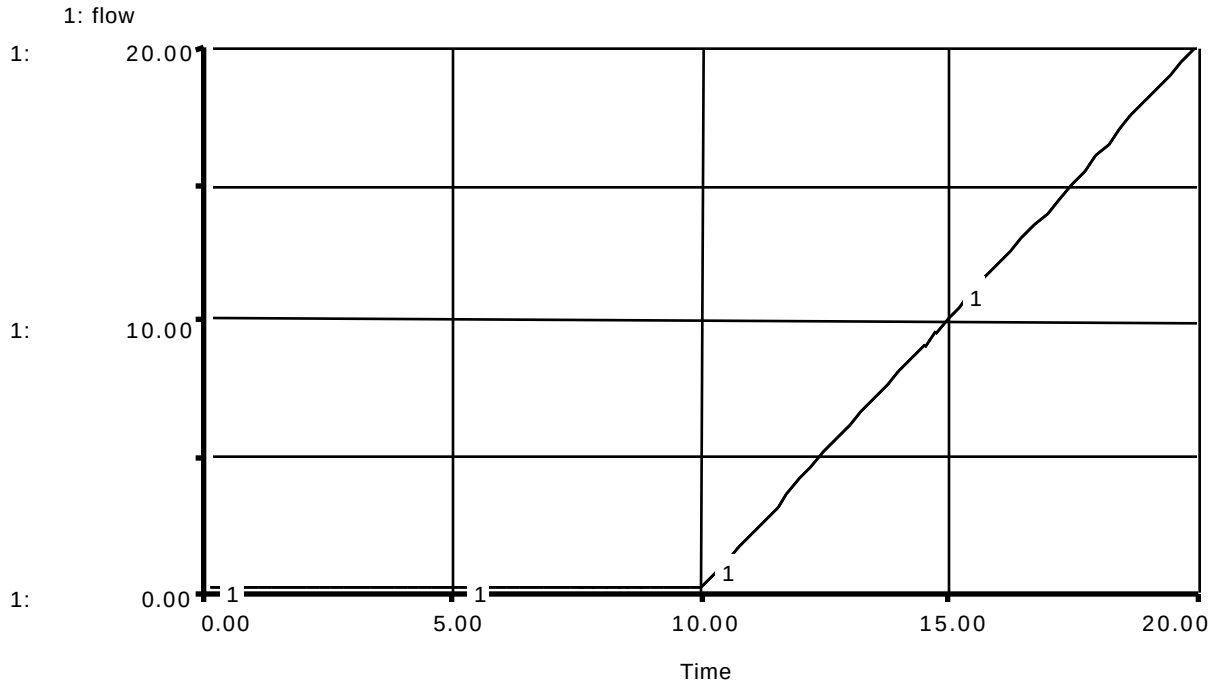


Figure 4: A linearly increasing flow beginning at Time 10.

Figure 5 shows the stock's behavior.

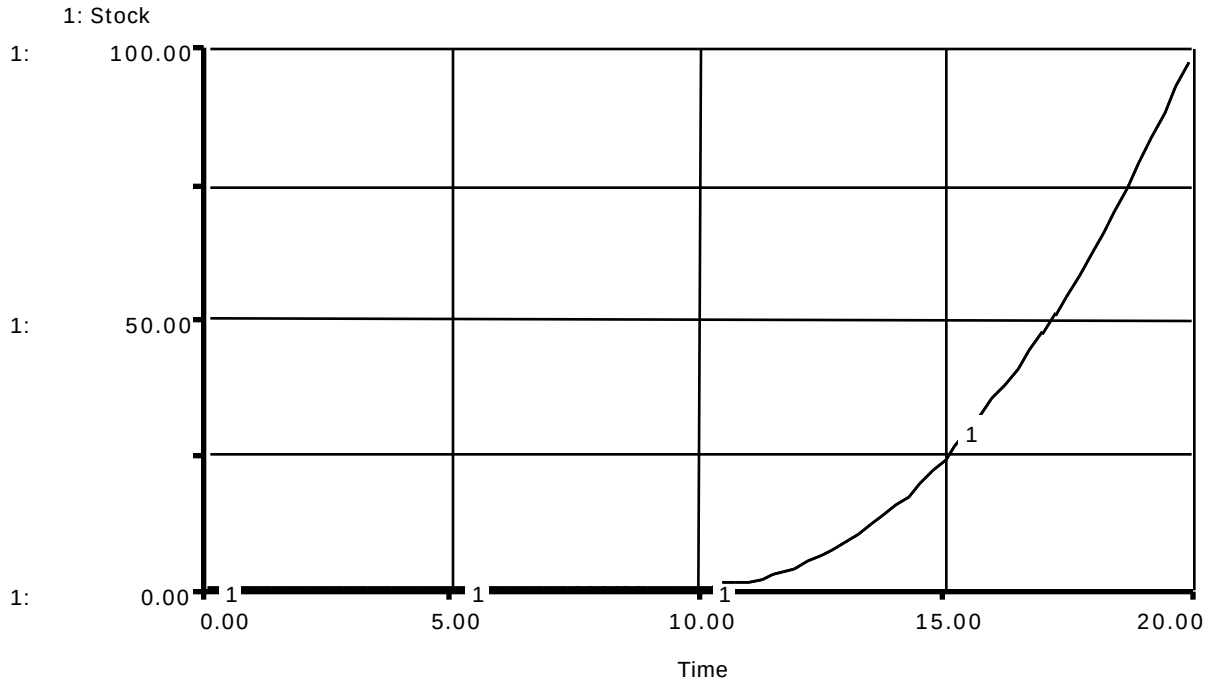


Figure 5: The behavior of the stock resulting from a linearly increasing flow beginning at Time 10.

The stock fills parabolically, beginning at Time 10. Before that time, because the net flow into the stock is zero, we do not add or subtract anything from the stock. At Time 15, the stock is being filled at a rate of 10 units per unit time. At Time 20, the stock is being filled at a net flow of 20 units per unit time. It is the linearly increasing flow that causes the parabolic growth. When we calculate the area under the entire flow graph, we see that this area is 100, which is the change in the value of the stock:

Total Area = Area under the flat line until Time 10 + Area under the triangle
Total Area = 0 + 0.5 * 20 * (20 - 10) = 100

Final Value of Stock = Initial Value of Stock + Total Area
Final Value of Stock = 0 + 100
Final Value of Stock = 100

For practice, try drawing in the tangent line at Time 15. Does the slope of this line equal 10 (which is the value of the flow at Time 15)? (Do not worry if the slope of the line you drew is a little off. Understanding the idea is what is important.)

3.2 Linearly Decreasing Flows Cause "Decreasing" Parabolic Growth

Now that we have looked at linearly increasing flows, let us look at a linearly decreasing flow filling an initially empty stock. Figure 6 shows the behavior of the flow, and Figure 7 shows the resulting behavior of the stock.

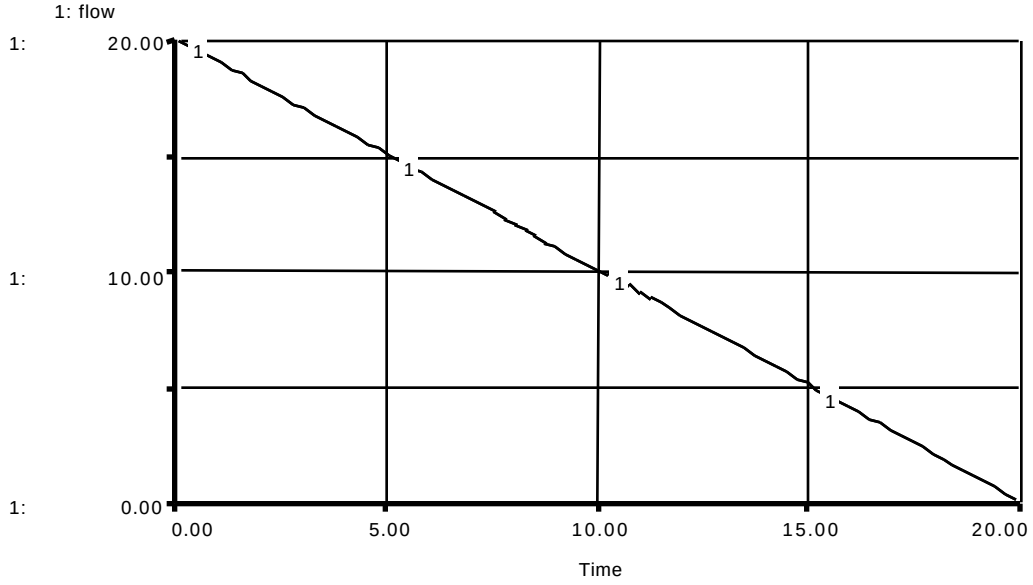


Figure 6: Linearly decreasing flow with negative slope.

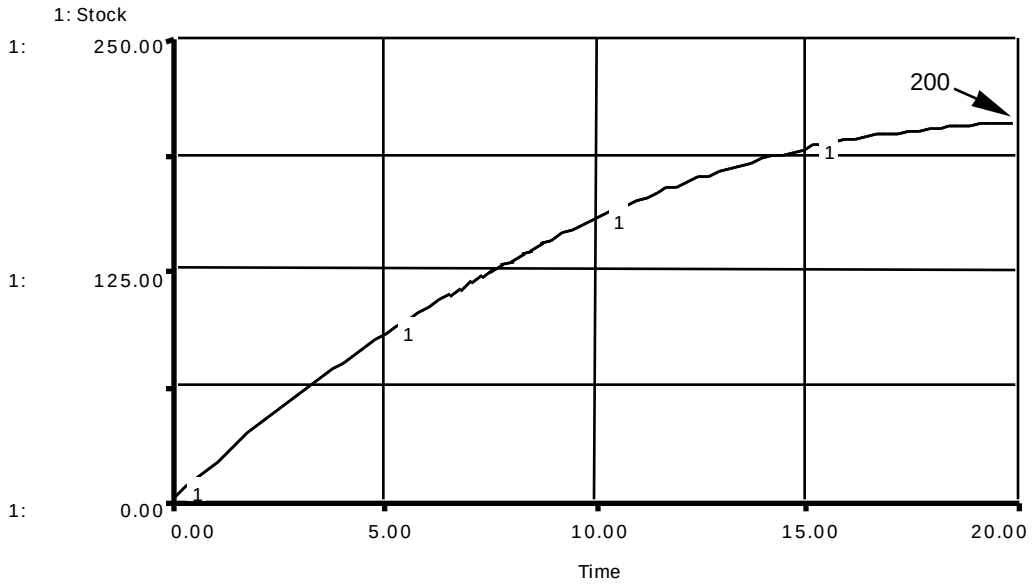


Figure 7: Behavior of Stock with a Linearly Decreasing flow.

As the flow value decreases, the rate of increase of the stock decreases. Notice that even though the flow value is decreasing, the level of the stock is still increasing. **Anytime there is a net positive inflow into a stock, the stock's value will increase.** At Time 0 the stock is filling at a rate of 20 units per unit time. At Time 5 the stock is still being filled, but only at the rate of 15 units per unit time. By Time 20 the filling rate is zero, and the stock remains constant. Thus, unlike systems with a linearly increasing flow that exhibit parabolic behavior, systems with downward sloping flows exhibit decreasing parabolic growth.

The stock is behaving as we expected because the change in the value of the stock is equal to the area under the flow graph.

Area under flow graph = 0.5 * 20 * 20 = 200.

This is equal to the change in the value of the stock:

Final Value of Stock = Initial Value of Stock + Change in the Value of Stock

Final Value of Stock = 0 + 200 = 200.

You can also check the proper behavior of the stock by drawing in tangent lines at various points. When we draw in tangent lines on the stock graph, we see that at the beginning the tangent lines are very steep. By Time 20, when the fill rate is zero, the tangent line is flat (zero slope). To prove this, we can draw two tangent lines into the graph shown in Figure 7.

Figure 8 shows the graph of the stock, the water level of the bathtub we discussed earlier, with tangent lines at Time 5 and Time 20.

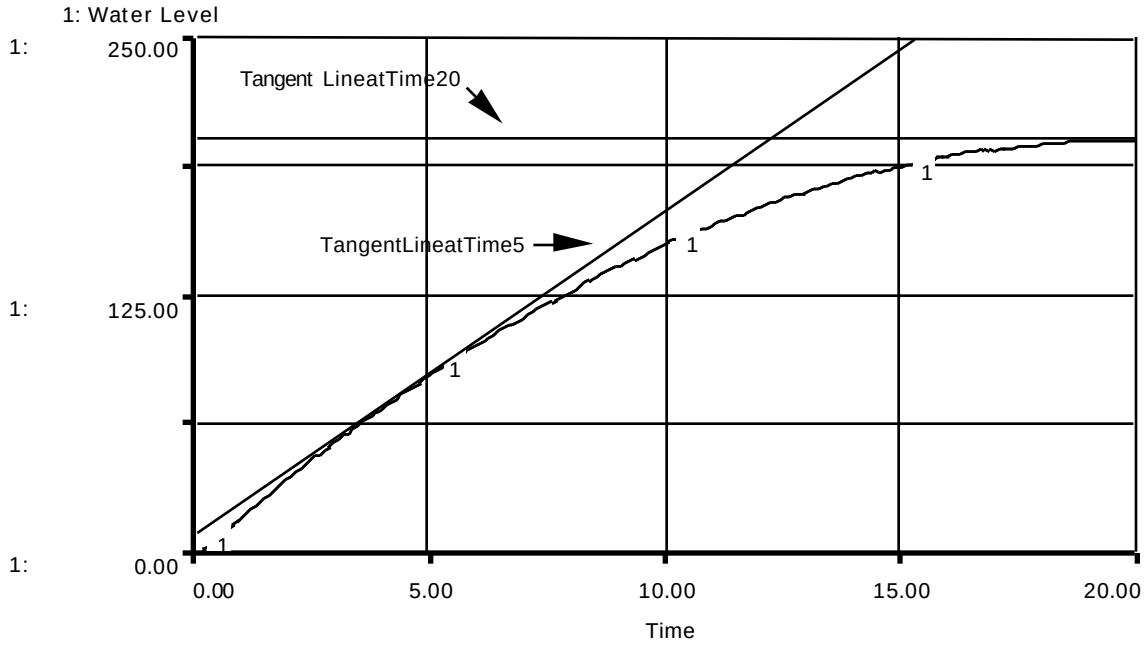


Figure 8: The result of a linearly decreasing flow filling an empty stock with the tangent lines drawn in at Time 5 and Time 20.

Let us look at another example of a linearly decreasing flow. We will again assume the initial value of the stock to be zero. Figure 9 shows the behavior of the flow and Figure 10 shows the resulting behavior of the stock.

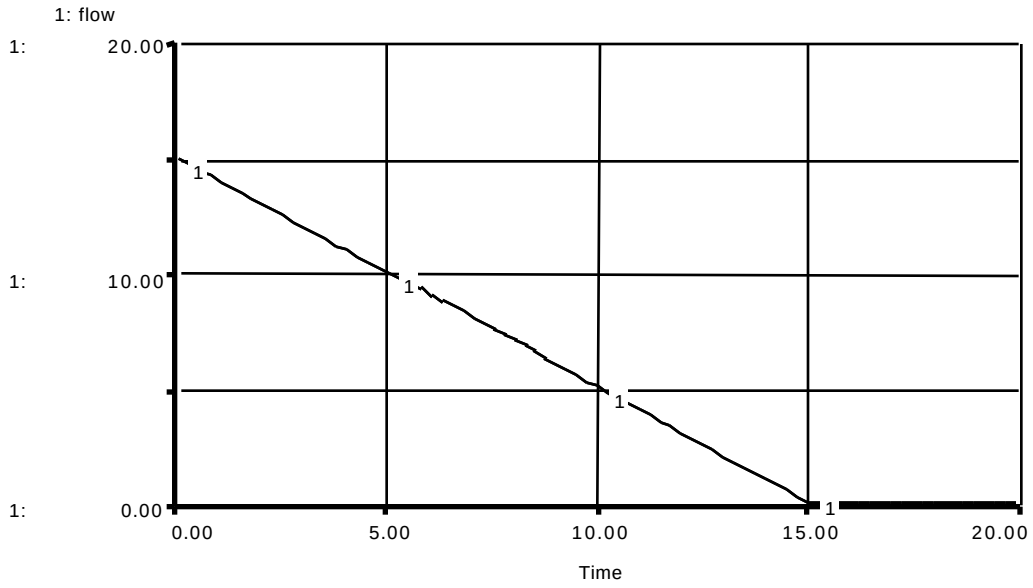


Figure 9: A linearly decreasing flow from Time 0 to Time 15.

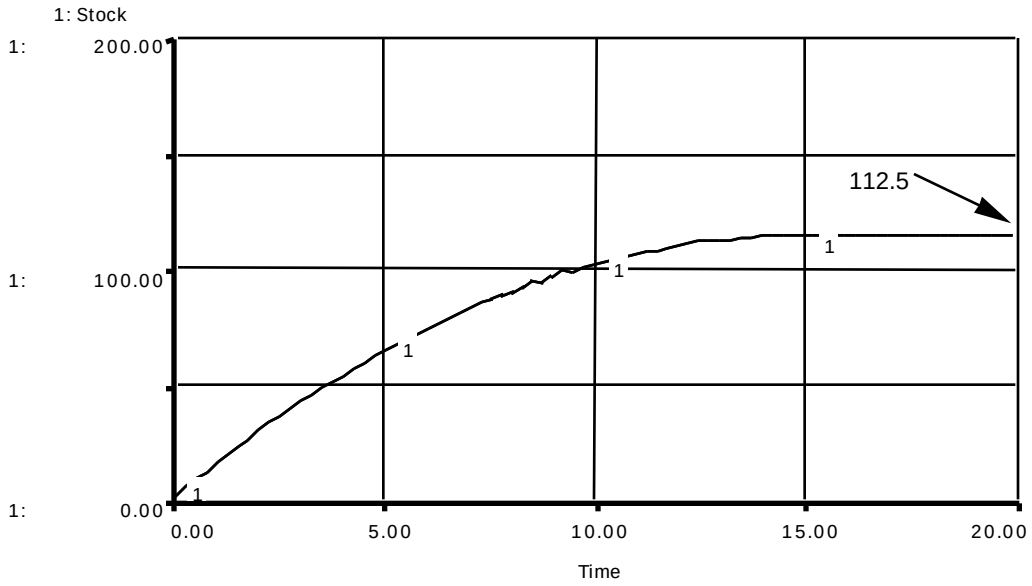


Figure 10: The stock resulting from a linearly decreasing flow from Time 0 to Time 15.

Just as we expected, as shown on Figure 10, the change in the value of the stock is equal to 112.5, which is the area under the flow graph. Also, the slope of the stock starts off steep and reaches flatness by Time 15. The flow into the stock becomes zero at Time 15, after which the level of the stock remains constant. For practice, try drawing the tangent line at Time 5. Does the slope of that line approximately equal 10?

4. Key Ideas

Now that we have seen how stocks behave when the flows are linearly increasing or decreasing, we are ready to try some exercises. Anytime you are performing graphical integration, keep the following rules in mind:

1. **When the net flow is positive, stocks are filled; when the net flow is negative, stocks are emptied.**
2. **The area under the flow graph over a period of time is equal to the change in the value of the stock over that same period of time. This helps us estimate the value of the stock if we know the initial value of the stock and the value of the flow:**
   Final Value of Stock = Initial Value of Stock + Area under flow graph
3. **The value of the flow gives the slope of the stock at that instant of time.**

4. **Constant flows cause the stock to increase linearly with the slope of the stock equal to the value of the flow.**[5]
5. **Linearly increasing flows cause the stock to exhibit parabolic growth.**
6. **Linearly decreasing flows cause the stocks to exhibit "decreasing" parabolic behavior.**
7. **You can often break up complicated graphs into several small, simpler ones.**

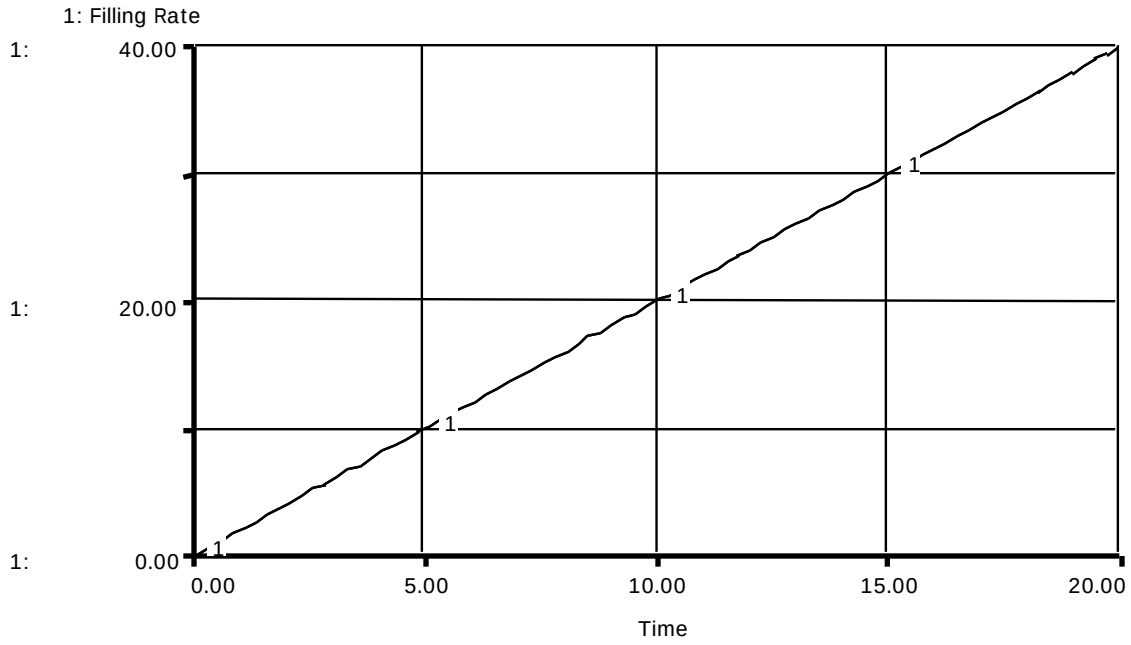
Now, let's try some exercises!!!

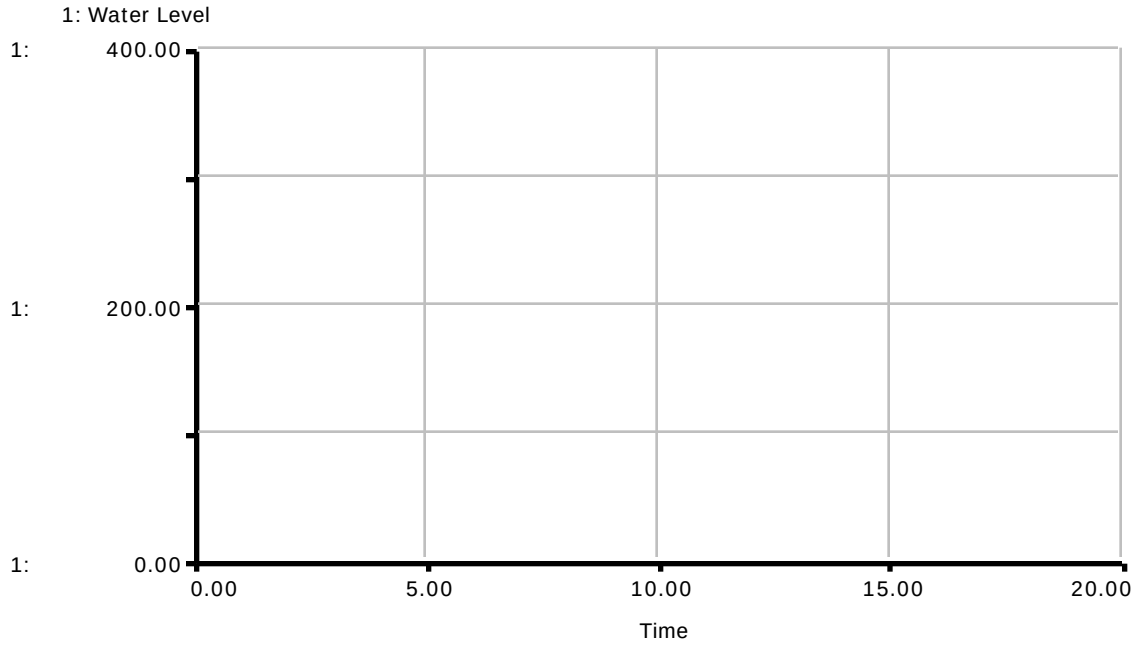
5. Exercises

5.1 Linearly Increasing Flow

In the first exercise, the flow of water filling a bathtub is initially 0 gallons per minute, and it is increasing linearly with a slope of +2 gallons per minute. Using the blank graph pad below, graph the behavior of the system. Assume the initial value of the stock is 0. (Hint: First figure out what type of behavior the stock will display, then calculate the area under the flow graph.)

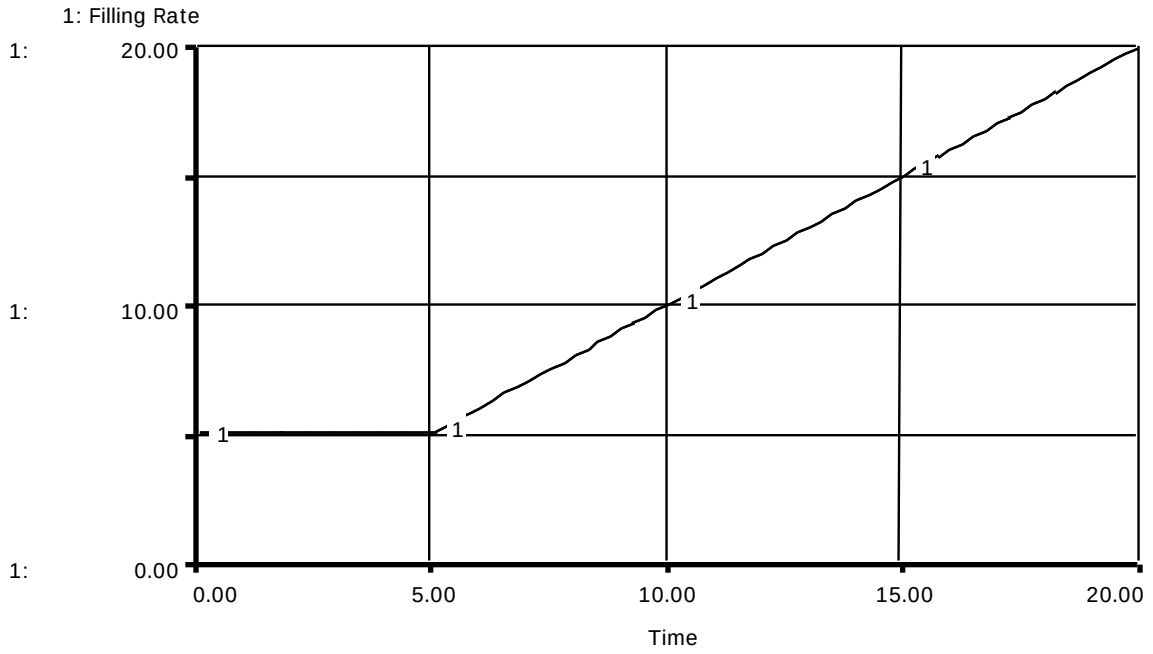
[5] If you are still unclear about how to graphically integrate systems with constant flows, please see: Alice Oh, 1995. *Graphical Integration Exercises Part One: Exogenous Rates* (D-4547), System Dynamics in Education Project, System Dynamics Group, Sloan School of Management, Massachusetts Institute of Technology, Dec. 1, 19pp.

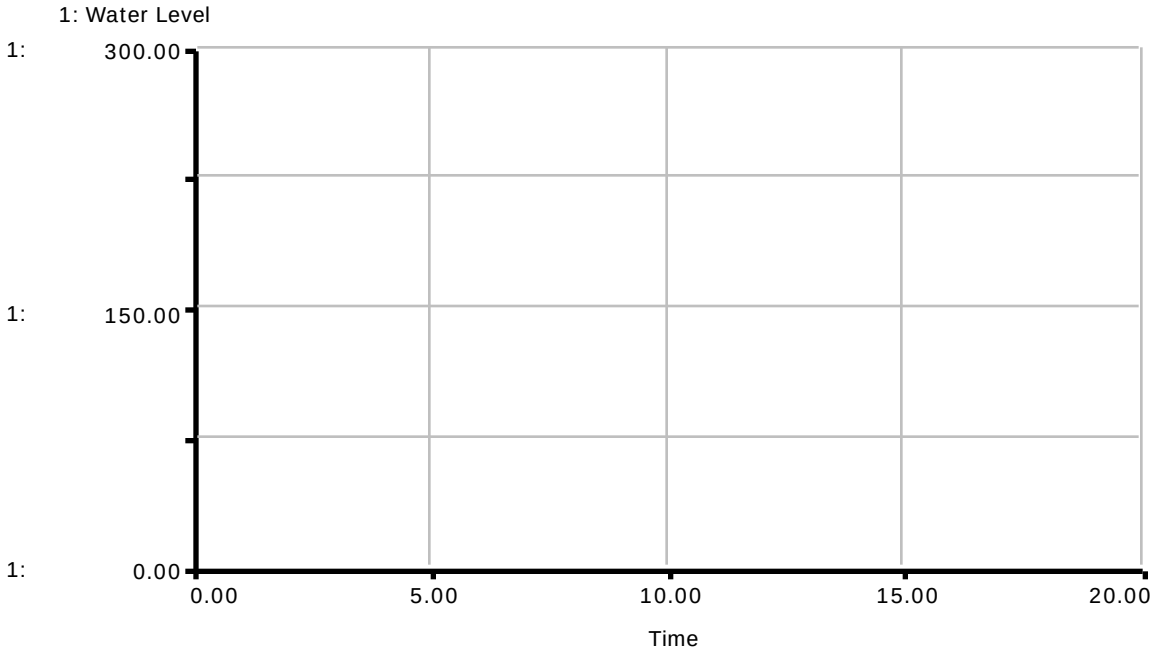




5.2 Constant and Linearly Increasing Flows

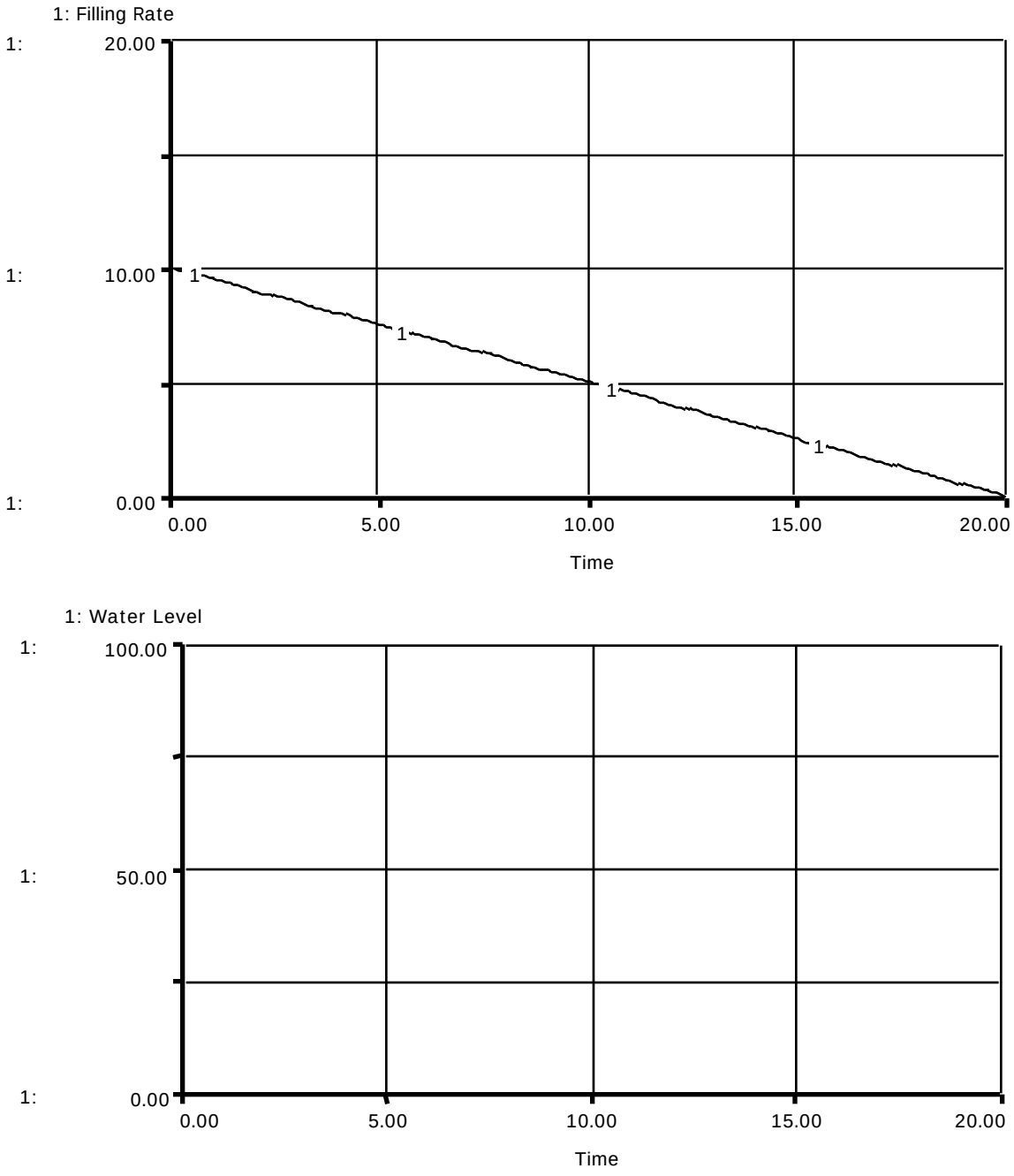
In this case, the rate of filling the bath tub is initially constant at +5 until Time 5. At Time 5, the flow increases linearly with a slope of +1. Assuming the initial value of the stock to be 75, graph its behavior below.





5.3 Linearly Decreasing Flow

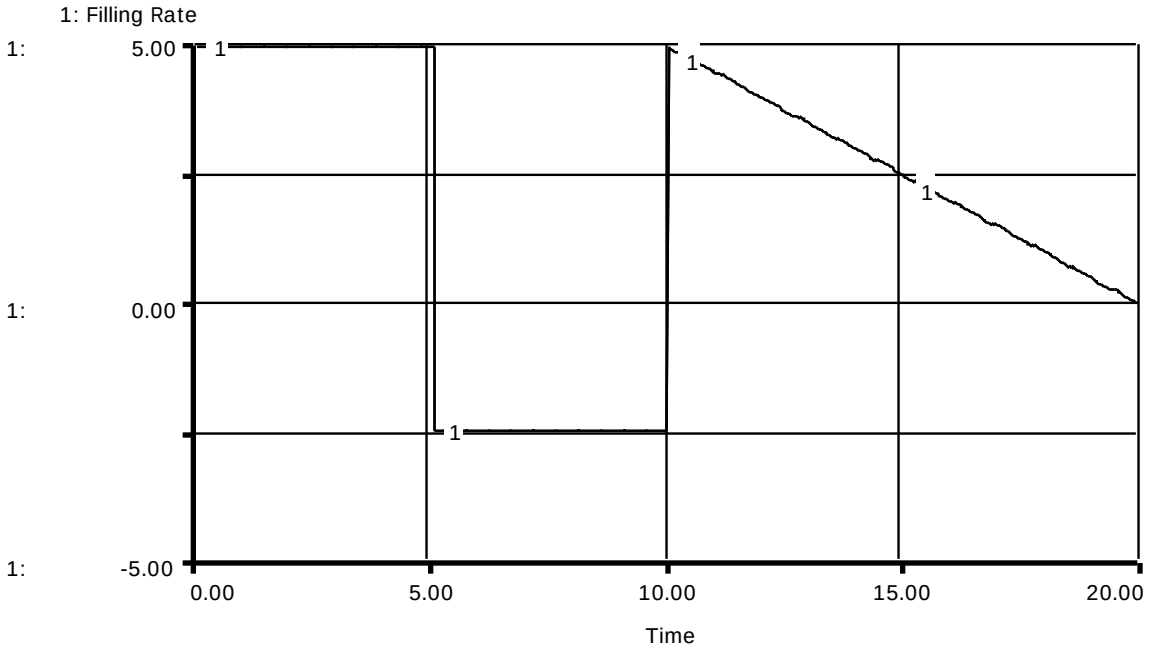
Now, let us try a linearly decreasing flow. The flow in this exercise starts at ten, and decreases to zero at Time 20. The initial value of the stock is zero.

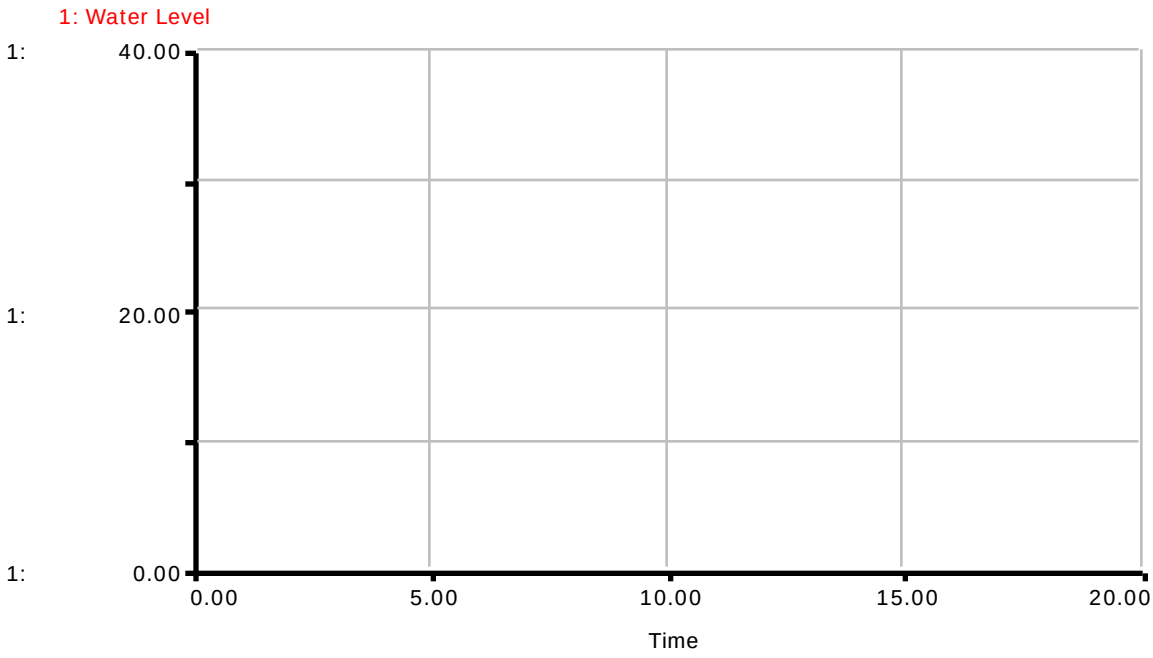


5.4 Constant and Linearly Decreasing Flows

Now let's try a more complex problem. From Time 0 to Time 5, the flow is constant at +5. At Time 5, the flow reverses to –2.5. At Time 10, the flow again turns positive and

steps up to 5 and linearly decreases to zero by Time 20. Assume the initial value of the stock is zero. (Hint: Break this problem up into three smaller problems.)[6]

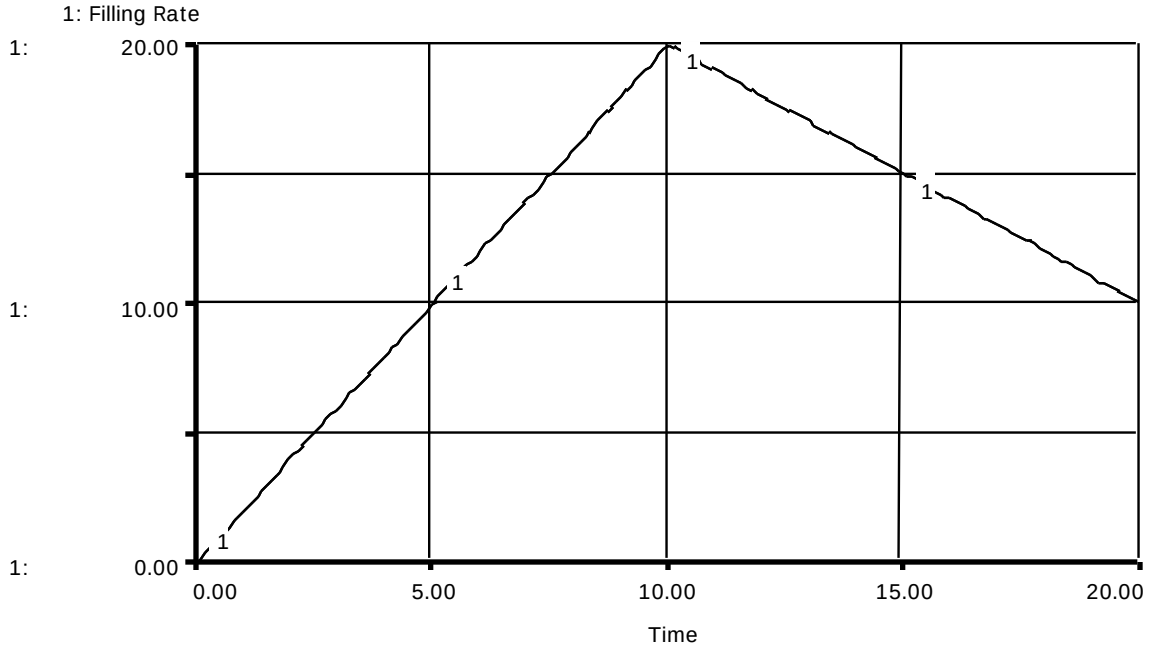


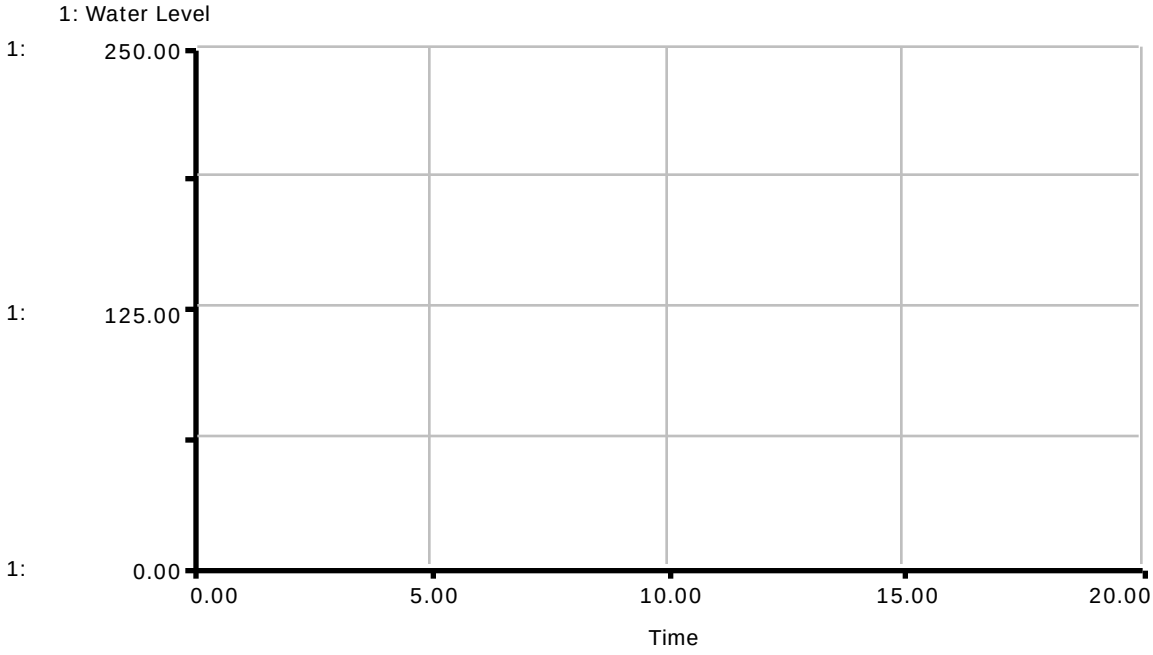


[6] If you feel uncomfortable dealing with negative flows, please review: Alice Oh, 1995. *Graphical Integration Exercises Part One: Exogenous Rates* (D-4547), System Dynamics in Education Project, System Dynamics Group, Sloan School of Management, Massachusetts Institute of Technology, Dec. 1, 19pp.

5.5 Linearly Increasing and Decreasing Flows

For our last exercise, let's try both upward and downward sloping flows. From Time 0 to Time 10, the flow is increasing from 0 to 20. From Time 10 to Time 20, the flow decreases from 20 to 10. Assume the initial value of the stock is 0. Graph the behavior of the stock below.





6. Conclusion

In this paper we looked at graphical integration with linearly increasing and decreasing flows. You should now feel comfortable using graphical integration as a method for estimating the behavior of systems with constant flows, step function flows, as well as linearly increasing and decreasing flows.

In the next paper of the Graphical Integration Exercises series, you will learn how to graphically integrate systems with separate inflows and outflows. You will need the skills you've acquired in this paper for the next in the series of papers on graphical integration.

7. Appendix: Answers to Exercises

7.1 Answer to Section 5.1

The water level, which is the stock, is simply a parabolic growth curve. Notice that the area under the flow graph is:

Area under flow graph = 0.5 * 40 * 20

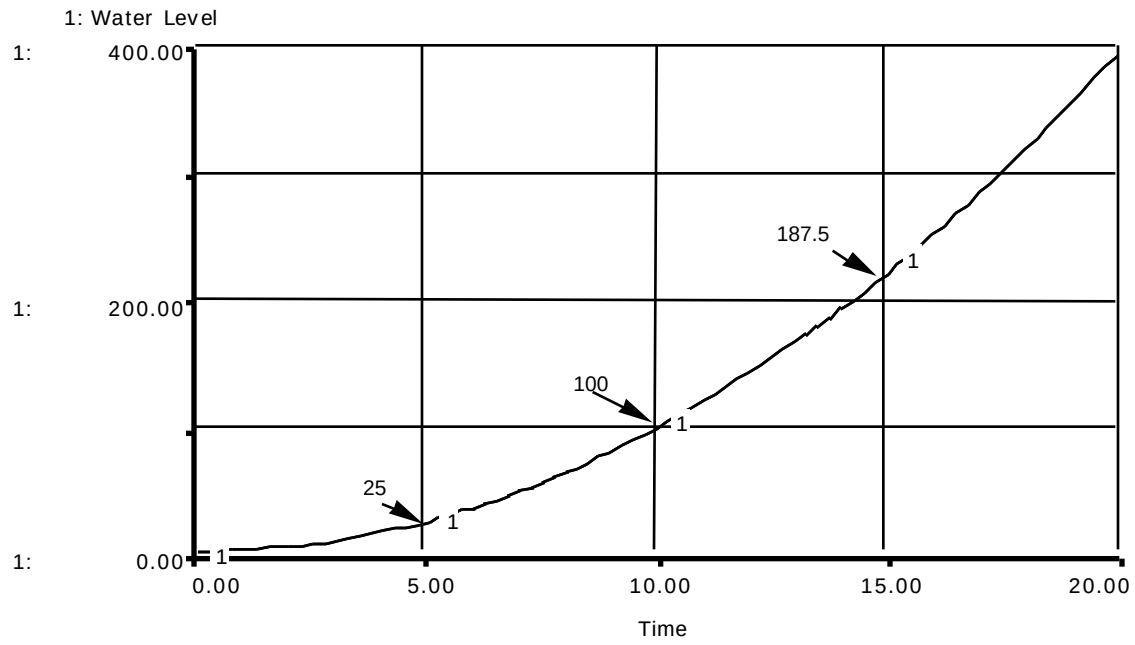
Area under flow graph = 400

This is equal to the change in the value of the stock:

Final Value of Stock = Initial Value of Stock + Change in the Value of Stock

Final Value of Stock = 0 + 400 = 400

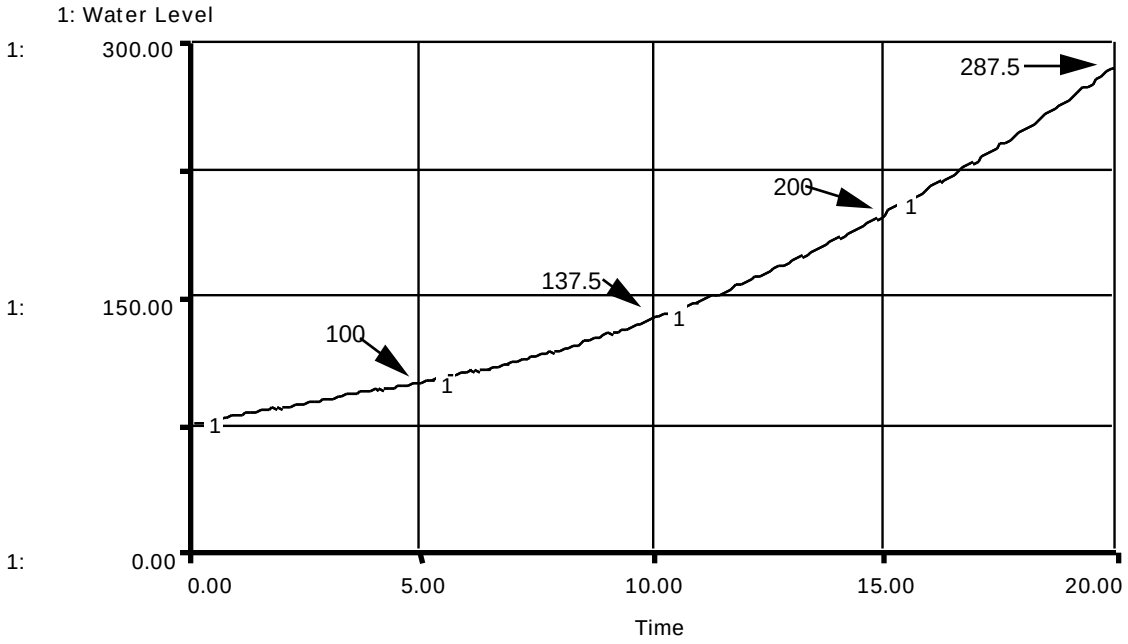
You could have also calculated the partial values of the stock by calculating the areas of the triangles and rectangles below the flow curve after each 5 units of time. Also notice that if you drew in a tangent line at Time 10 its slope would be 20, which is the filling rate at Time 10.



7.2 Answer to Section 5.2

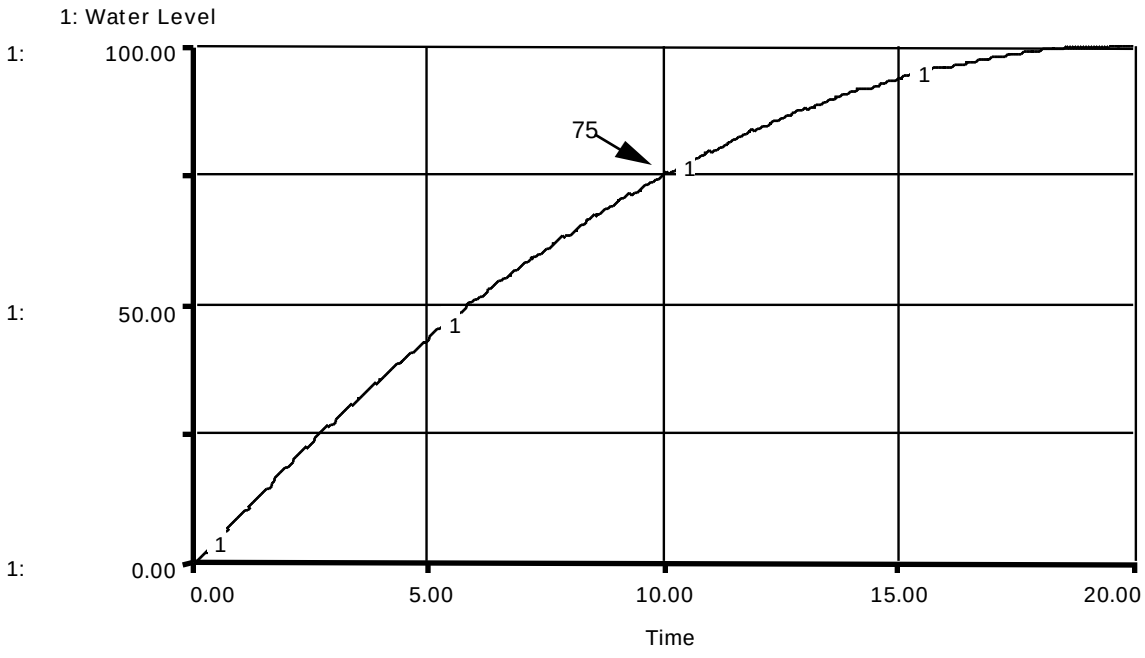
The second problem can be broken into two parts. The first part is from Time 0 to Time 5, when the flow is constant. Notice that the first part of the curve is linear, with a slope of 5, which is the constant flow value. Also notice that the area under the flow graph for this time period is 5 times 5, or 25.[7] This area is the change in the value of the stock. Because the stock was initially 75, at Time 5, the stock's new value is 75 + 25, or 100. For Time 5 to Time 20, the stock undergoes parabolic growth because of the ramping flow. To calculate the change in the level of the stock, we need to calculate the area under the flow graph from Time 5 to Time 20. This is most easily done by breaking the area under the flow graph into a rectangle and a triangle. The rectangle has a length of 15 and a height of 5. Thus its area is 75. The base of the triangle is 15, and its height is 15, so its area is 112.5. Thus the total area under the flow graph from Time 5 to Time 20 is 75 + 112.5 = 187.5. Because the stock, as we determined, was at 100 at Time 5, the value of the stock at Time 20 is 100 + 187.5, or 287.5.

[7] The area of a rectangle is simply its length times its height.



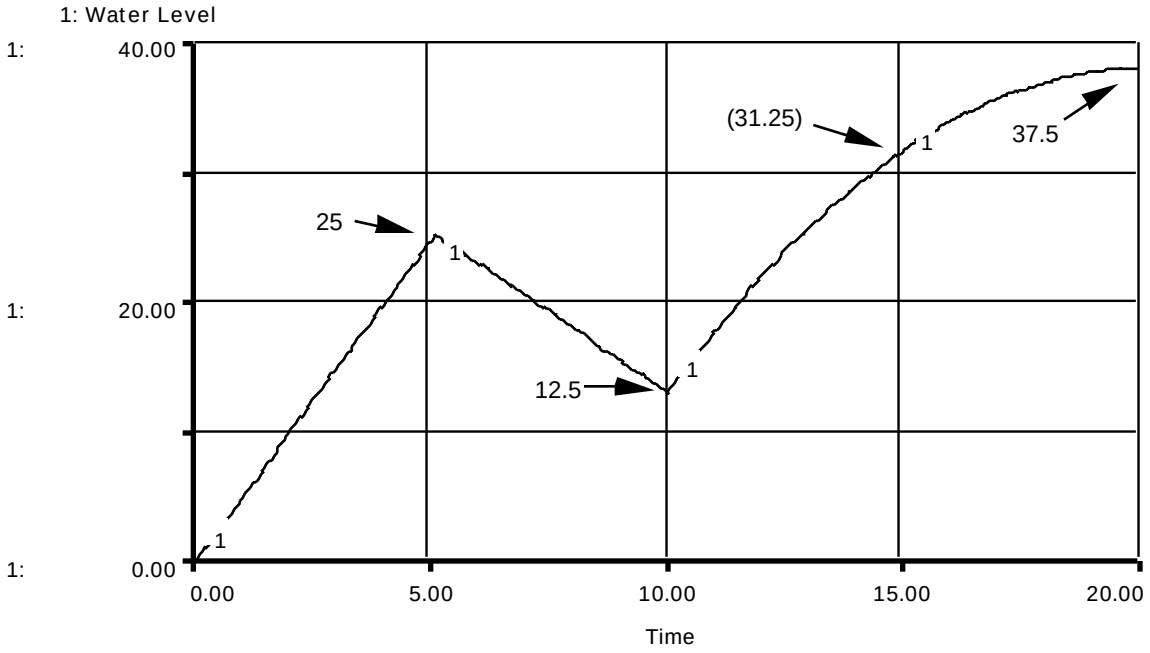
7.3 Answer to Section 5.3

Because the flow is simply a downward sloping line, the stock will undergo a downward parabolic behavior. Initially it is increasing very rapidly. As time continues, the value of increase gets smaller and smaller, equaling zero at Time 20. The area under the flow graph is 0.5 times 20 times 10, which equals 100. Because the stock was initially empty, the final value of the stock is 0 + 100, or 100.

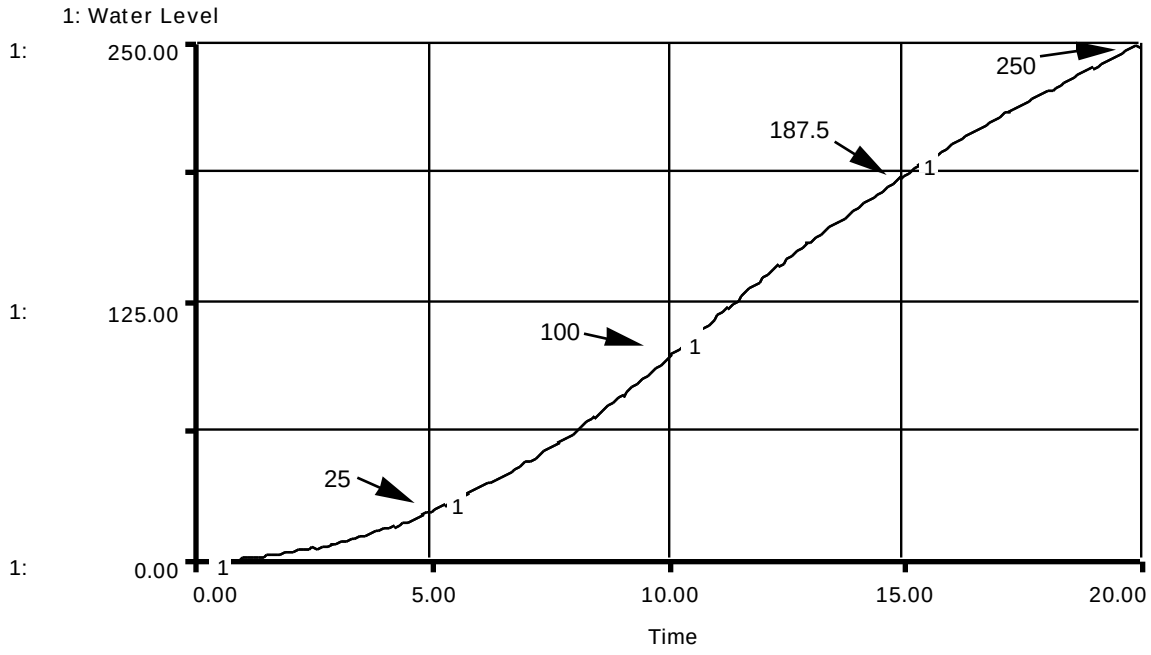


7.4 Answer to Section 5.4

To solve this problem, it is easiest to break it up into three parts: from Time 0 to Time 5, from Time 5 to Time 10, and from Time 10 to Time 20. For the first part, from Time 0 to Time 5, the stock increases linearly because the flow is a constant positive number. To calculate the value of the stock at Time 5, we only need to find the area under the flow graph from Time 0 to Time 5. This area is simply 5 times 5, or 25. Because the initial value of the stock was zero, the value of the stock at Time 5 is 25. For Time 5 to Time 10, the stock decreases linearly because the constant flow is negative. Again, calculating the area between the flow graph and the zero flow line, we see that the stock decreased by 12.5, yielding a value of the stock at Time 10 to be 25 - 12.5, or 12.5. For the third part, the stock displays a downward parabolic behavior, because the flow is ramping downwards. The area under the flow graph is 0.5 times 10 times 5, or 25. Thus, the value of the stock at Time 20 is the value of the stock at Time 10 plus 25. Because the value of the stock at Time 10 was 12.5, the value of the stock at Time 20 is 12.5 + 25 = 37.5.



7.5 Answer to Section 5.5

Again, it is easiest to answer the last problem by separating it into two parts. The first part, where the flow is ramping upwards, displays parabolic growth. The value of the stock at Time 10 is calculated by finding the area under the flow graph, which is 0.5

times 10 times 20. This equals 100. The second part, which begins at Time 10, displays a downward parabolic behavior because the flow is sloping downward. The area under the flow graph for this period can be broken up into the area of the rectangle with a length of 10 and a height of 10, plus the area of the triangle with a base of 10 and a height of 10. The sum of these two areas is 100 + 50 = 150. Because the total area under the flow graph is the change in the value of the stock, the value of the stock at Time 20 is 100 (the value of the stock at Time 10) plus 150. Therefore, at Time 20, the value of the stock is 250.