16.001 Unified Engineering
Materials and Structures

Lecture M8-M9
Statically Indeterminate Systems

Reading assignments: Connor's: Ch. 2, CDL: Ch. 2, 2.3 2.4

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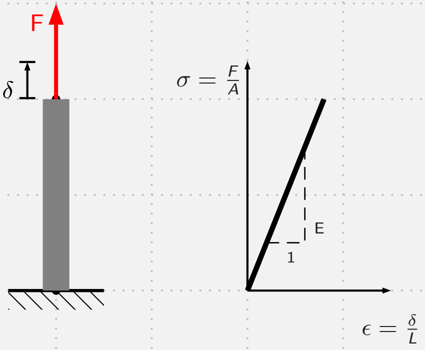
Outline

Material test



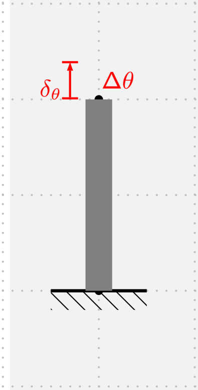
L: length of the bar
A: area of the cross section of the bar
E: material's Young's modulus
δ_m: elongation of the bar due to mechanical loading
F: applied force

Clearly, from here we get:

$$\boxed{F = \frac{EA}{L}\delta_m}$$

Deformation of bars due to temperature changes: Thermal strains

Temperature changes



L: length of the bar
α: material's coefficient of thermal expansion
δ_θ: elongation of the bar due to thermal expansion
F: applied force

A temperature change $\Delta\theta$ causes an elongation (strain, not stress or force) that is proportional to the length of the bar L and to $\Delta\theta$. Temperature increases $\Delta\theta > 0$ produces en expansion $\delta_\theta > 0$ and viceversa. Then:

$$\boxed{\delta_\theta = \alpha \Delta\theta L}$$

$$\text{Dimensions of } \alpha : [\delta_\theta] = \cancel{L} = [\alpha]\Theta\cancel{L}, \rightarrow [\alpha] = \Theta^{-1}$$

In the SI, the units of α will be K^{-1}. Typical values of α are: $\alpha_{\text{aluminum}} = 23 \times 10^{-6} K^{-1}, \alpha_{\text{steel}} = 13 \times 10^{-6} K^{-1}, \alpha_{\text{titanium}} = 9 \times 10^{-6} K^{-1}$.

Remember

Temperature changes cause deformations (strains), not stresses. If these deformations are constrained, then "thermal stresses" arise.

Deformation of bars due to combined temperature changes and loads

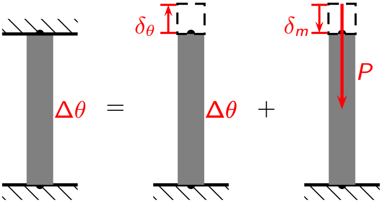
Combined temperature and force loading of bars

By superposition, the combined loading will produce a total elongation δ_T which is the sum of the elongation due to the applied force or mechanical elongation δ_m, and the elongation produced by thermal expansion or thermal elongation δ_θ

$$\delta_T = \delta_m + \delta_\theta$$

$$\boxed{\delta_T = \underbrace{\frac{PL}{EA}}_{\delta_m} + \underbrace{\alpha \Delta\theta L}_{\delta_\theta}}$$

Example: Built-in bar subject to temperature change



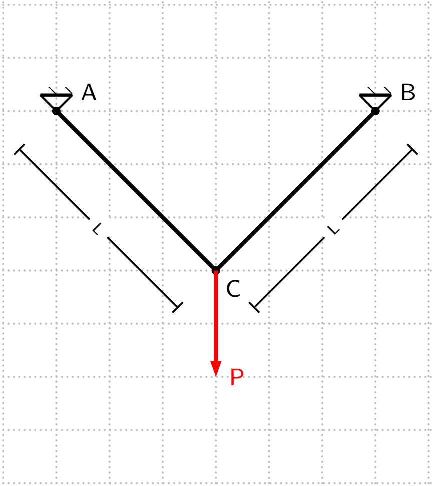
Use superposition:

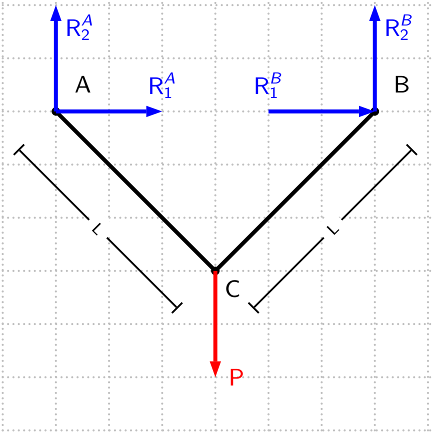
$$\delta_T = 0 = \delta_m + \delta_\theta, \rightarrow \delta_m = -\delta_\theta$$

$$\frac{P\cancel{L}}{EA} = -\alpha\Delta\theta\cancel{L}, \rightarrow \boxed{P = -\alpha EA\Delta\theta}$$

i.e.[5] a temperature increase (decrease) causes a compressive (tensile) force.

Objective: Compute the displacements of any joint in the truss.

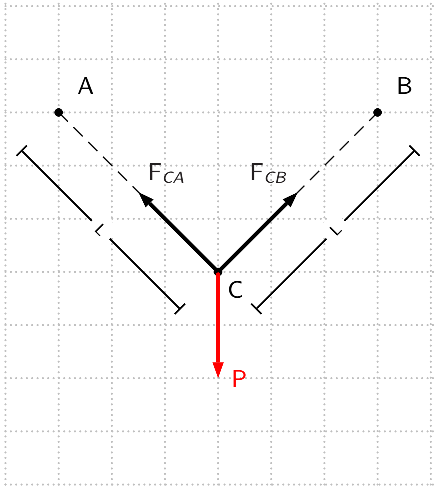


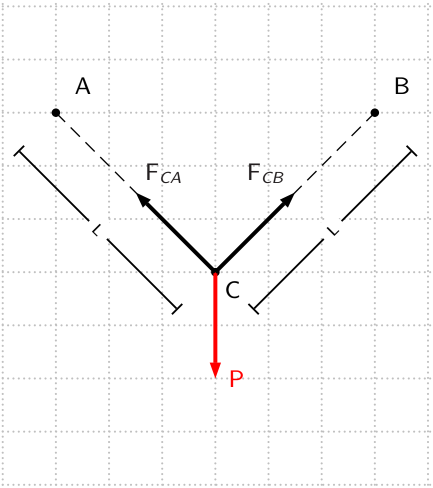


Equilibrium

- Note that the individual reaction components at each pin joint are not independent, as their resultant needs to be aligned with the bar. This is required as the bar can only carry axial forces.
- The reaction at the supports will therefore be of the same magnitude as the internal force in the respective bar, with an opposite direction.
- It suffices then to find out the internal loads on the bars, which can be done by applying method of joints to joint C.

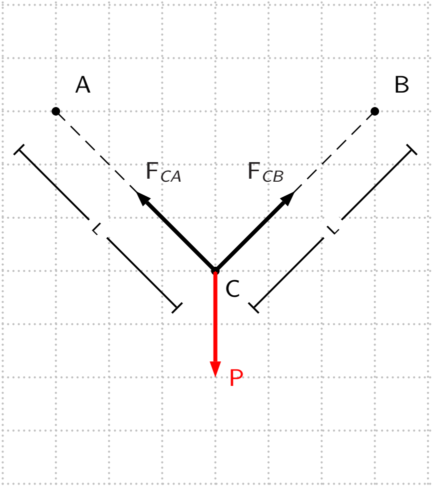
Method of Joints to determine force in each rod





Equilibrium of joint C

- ΣF_1: $-\frac{\sqrt{2}}{2}F_{CA} + \frac{\sqrt{2}}{2}F_{CB} = 0$
- ΣF_2: $\frac{\sqrt{2}}{2}F_{CA} + \frac{\sqrt{2}}{2}F_{CB} - P = 0$



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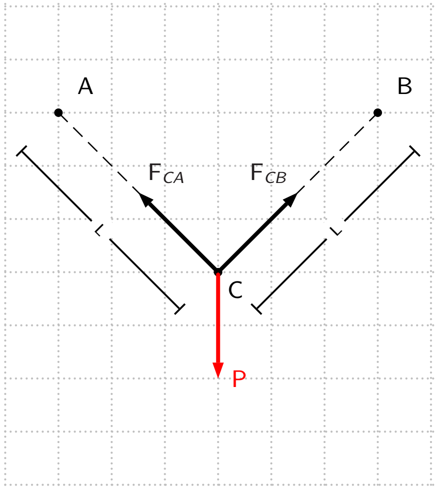
Forces in rods:

- From ΣF_1: $F_{CA} = F_{CB}$
- From ΣF_2: $F_{CA} = \frac{P}{\sqrt{2}}$



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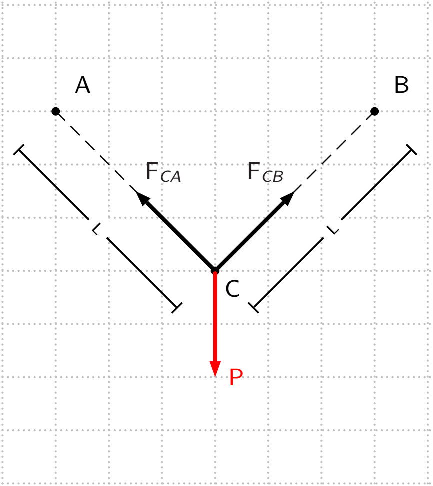

Equilibrium of joint C

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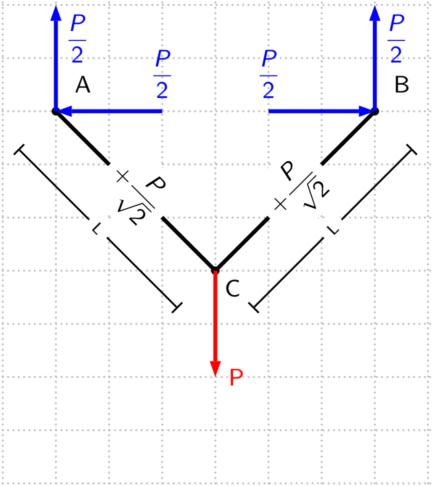
Forces in rods:

- From ΣF_1: $F_{CA} = F_{CB}$
- From ΣF_2: $F_{CA} = \frac{P}{\sqrt{2}}$

Having figured out the internal forces independently, we can compute the deformation of each bar and after the overall displacement of joint C

Constitutive relation to determine the deformation of each rod



Let's keep it simple for now and make the following assumptions for each rod:

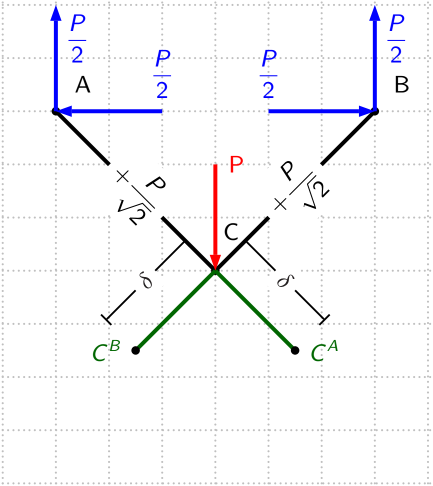
- Same Young's modulus, E
- Same cross section, A
- Stiffness: $K = \dfrac{EA}{L}$

Constitutive relation:

- between force F_{XY} and elongation δ
- $F_{XY} = K\delta \quad \Rightarrow \quad \delta = \dfrac{LP}{\sqrt{2}EA}$

Next, compatibility: compute displacement of joint C.

Constitutive relation to determine the deformation of each rod



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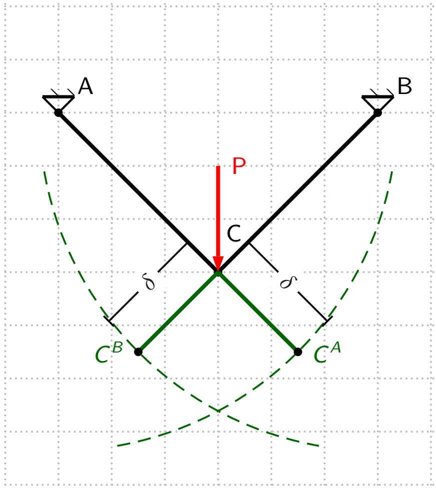
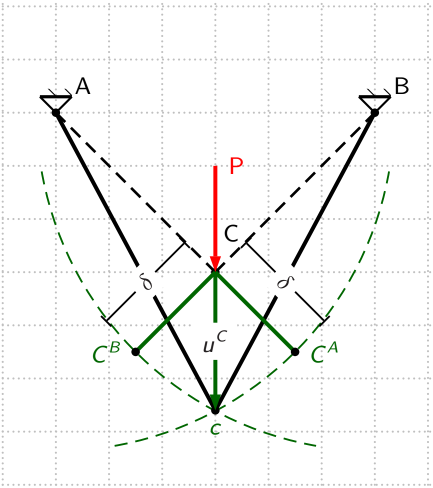
Compatibility to compute the displacement of joint C



Compatibility relations:

- $C \in$ Circle(A, L+δ)
- $C \in$ Circle(B, L+δ)

Compatibility to compute the displacement of joint C



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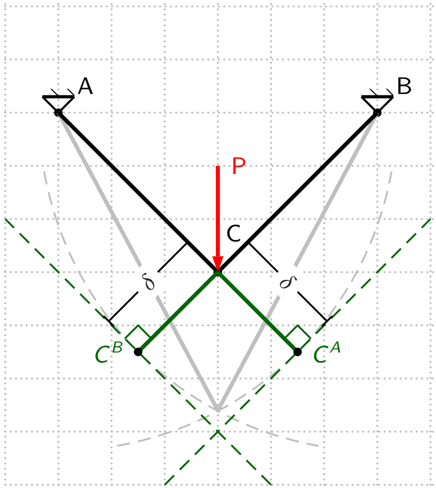

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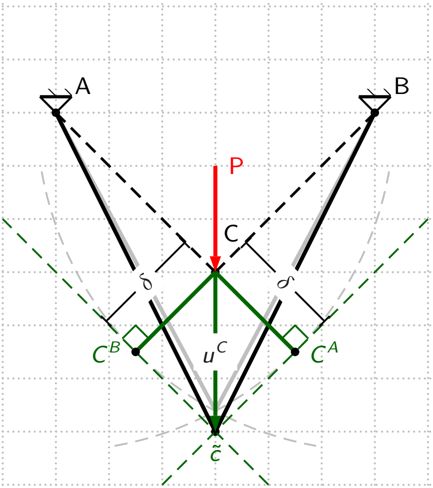
Small displacements and small rotations:

- Circle(A, L+δ) $\approx$ Tangent @C^A
- Circle(B, L+δ) $\approx$ Tangent @C^B

Compatibility to compute the displacement of joint C



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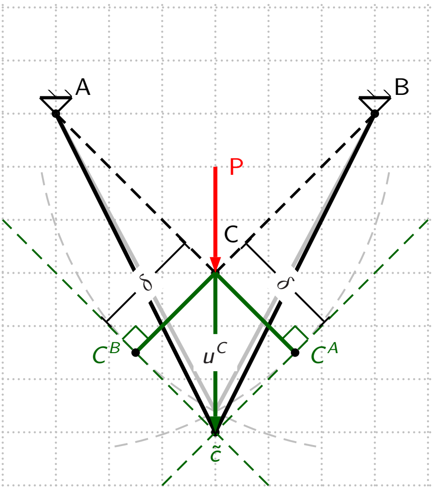

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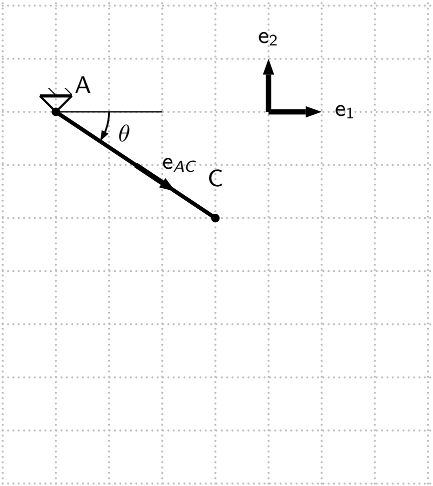
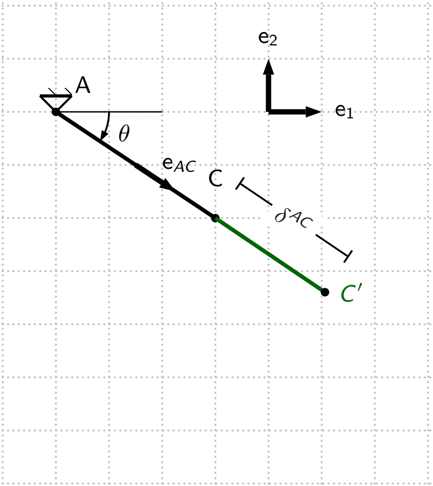
How to relate δ and u^C?

How to relate rod deformation δ^{AC} to joint displacement u^C



- Undeformed bar AC and basis vectors: $e_{AC} = \cos\theta e_1 - \sin\theta e_2$

How to relate rod deformation δ^{AC} to joint displacement u^C



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- Elongation (greatly exaggerated)

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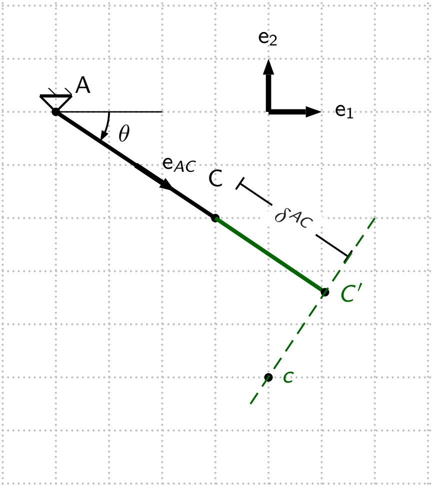
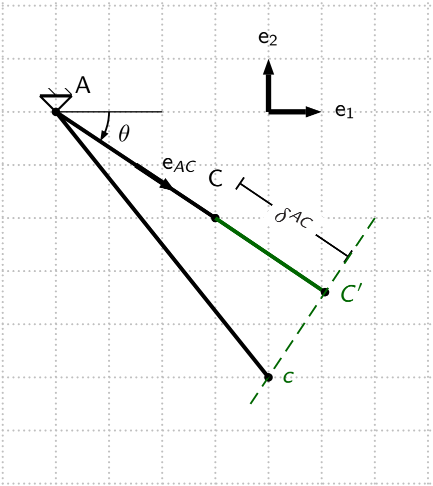


- Undeformed bar AC and basis vectors: $e_{AC} = \cos\theta e_1 - \sin\theta e_2$
- Elongation (greatly exaggerated)
- Deformed bar AC' can rotate around point A, circular motion approximated by tangent $C'c$.



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- Deformed bar Ac

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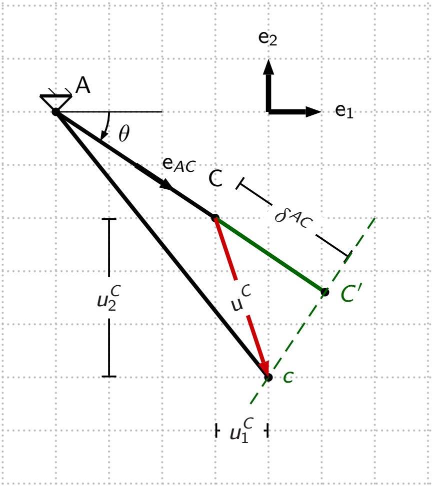

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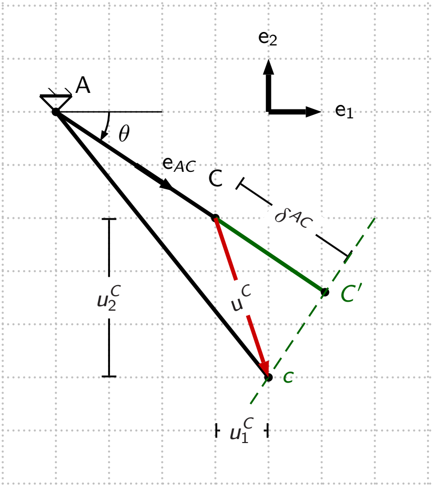
- Displacement vector $\mathrm{u}^C = u_1^C \mathrm{e}_1 - u_2^C \mathrm{e}_2$



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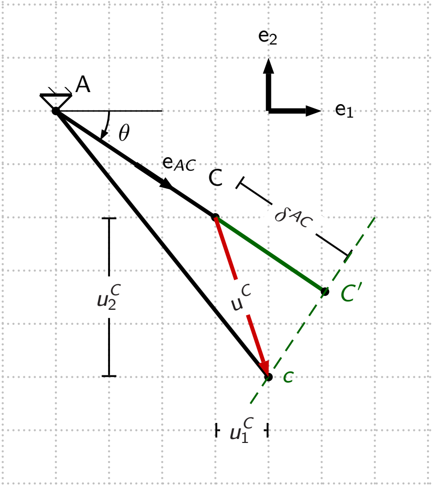
- Displacement vector $u^C = u_1^C e_1 - u_2^C e_2$
- Obtain sought relation by noticing that the elongation δ^{AC} is ALWAYS the projection of the displacement vector on the undeformed direction of the bar.

How to relate rod deformation δ^{AC} to joint displacement u^C



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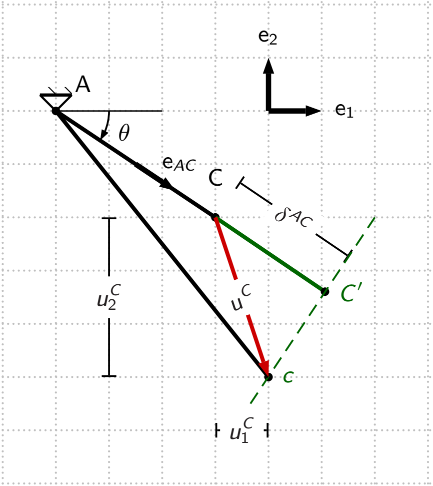
$$\mathrm{u}^C \cdot \mathrm{e}_{AC} = \delta^{AC}$$

$$u_1^C \cos\theta + (-u_2^C)(-\sin\theta) = \delta^{AC}$$

$$\boxed{u_1^C \cos\theta + u_2^C \sin\theta = \delta^{AC}}$$

How to relate rod deformation δ^{AC} to joint displacement u^C



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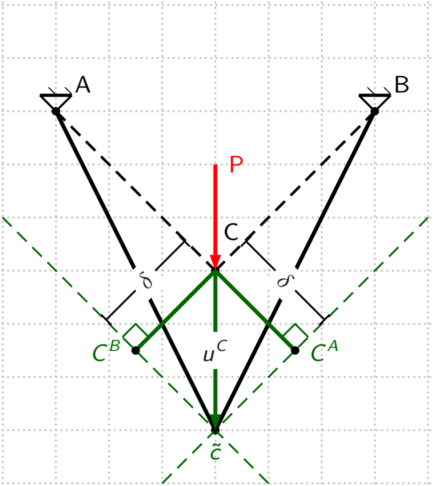
$$u^C \cdot e_{AC} = \delta^{AC}$$

$$u_1^C \cos\theta + (-u_2^C)(-\sin\theta) = \delta^{AC}$$

$$\boxed{u_1^C \cos\theta + u_2^C \sin\theta = \delta^{AC}}$$

For each bar, compatibility equation relating elongation with joint displacement



From constitutive relation:

$$\delta = \frac{LP}{\sqrt{2}EA}$$

From displacement compatibility:

$$\delta = u_1^C \cos\theta + u_2^C \sin\theta$$

From symmetry:

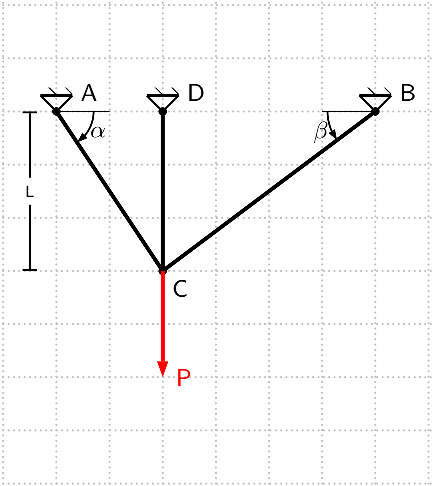
$$u_1^C = 0 \Rightarrow \delta = u_2^C \frac{\sqrt{2}}{2}$$

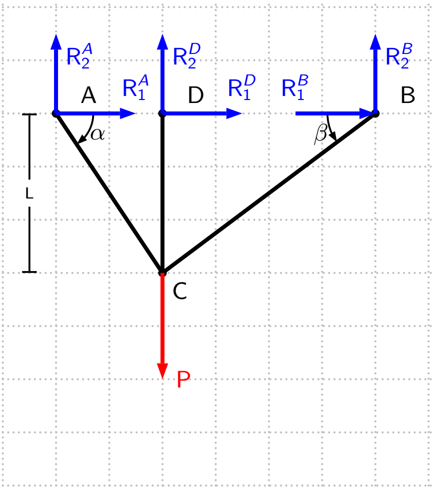
Then:

$$u_2^C = \frac{LP}{2EA}$$

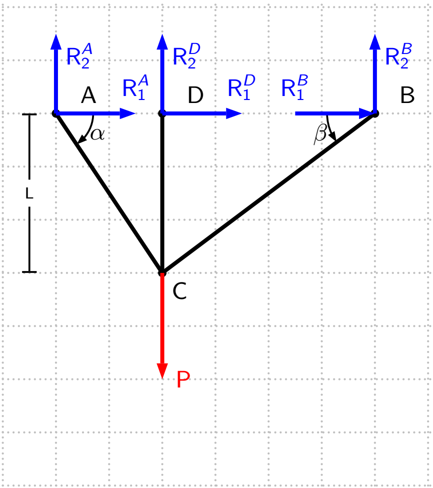
Use a very similar approach when solving statically indeterminate systems.

Objective: Compute internal forces in bars and displacements of joints.

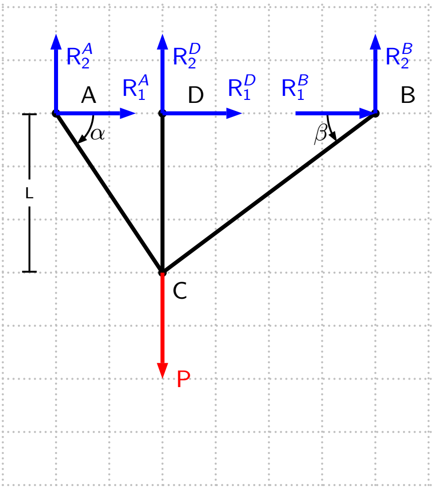




R_2^A
R_2^D
R_2^B
R_1^A
R_1^D
R_1^B
A
D
B
α
β
L
C
P

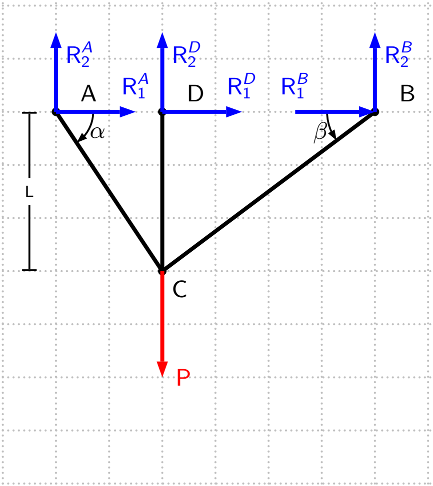


Notice that in this case the FBD of the overall structure does not help much: it exposes six unknown reaction components but we know that the reactions will need to be aligned with the corresponding bars and have the same magnitude and opposite direction, i.e. the reactions are fully determined once the forces on the bars are known. This means that we could skip the computation of the reactions at this point and concentrate on the bars. For completeness, we provide the equations.



Equilibrium

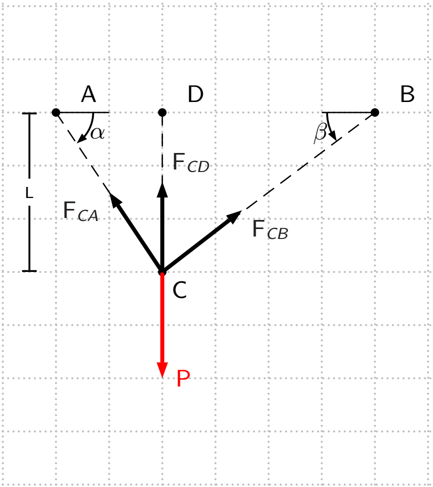
- ΣF_1: $R_1^A + R_1^D + R_1^B = 0$
- ΣF_2: $R_2^A + R_2^D + R_2^B - P = 0$
- ΣM^D:
  $$-\frac{L}{|\tan\alpha|}R_2^A + \frac{L}{|\tan\beta|}R_2^B = 0$$

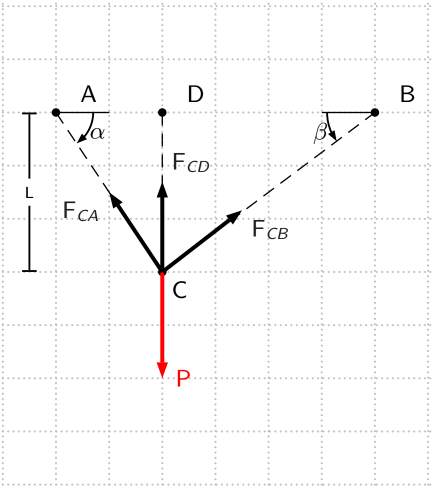


Reactions:

- In principle 6 unknowns & 3 equations
- However, members can only carry axial force: components of each reaction force vector are related (i.e. vector direction known, only magnitude is unknown, only three unknowns
- Another however: not really 3 equations since all reactions concurrent to point C, still indeterminate)

Method of Joints to determine force in each rod

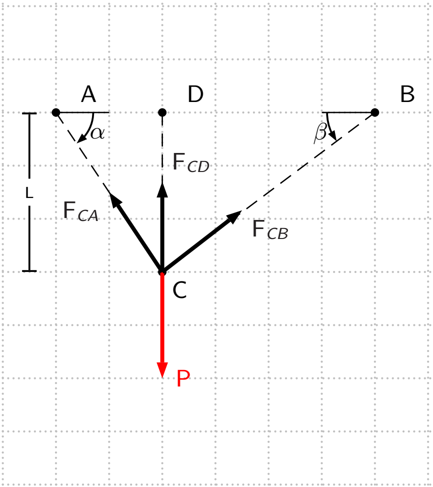




Equilibrium of joint C

- ΣF_1: $\cos\beta F_{CB} - \cos\alpha F_{CA} = 0$
- ΣF_2:
  $+\sin\beta F_{CB} + \sin\alpha F_{CA} + F_{CD} = P$

Equilibrium gives two equations and three unknowns, the problem is STATICALLY INDETERMINATE



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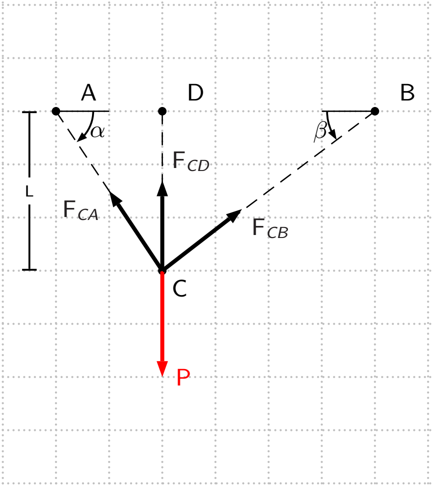

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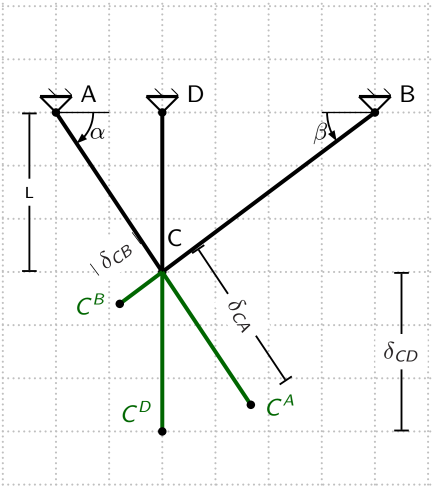
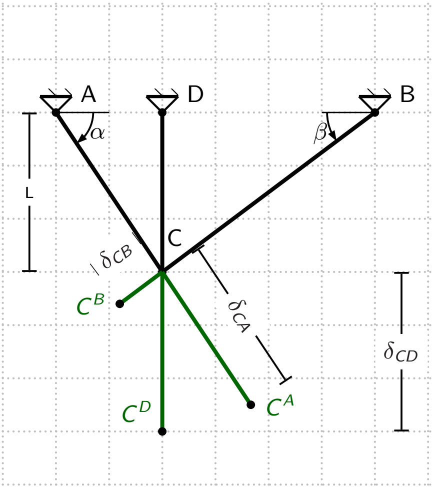
We need to consider the deformation of each rod

Constitutive relation to determine the deformation of each rod



Constitutive relation: for simplicity, we will assume:

- Same Young's modulus, E
- Same cross section, A
- $\delta_{XY} = \dfrac{L_{XY} F_{XY}}{EA}$



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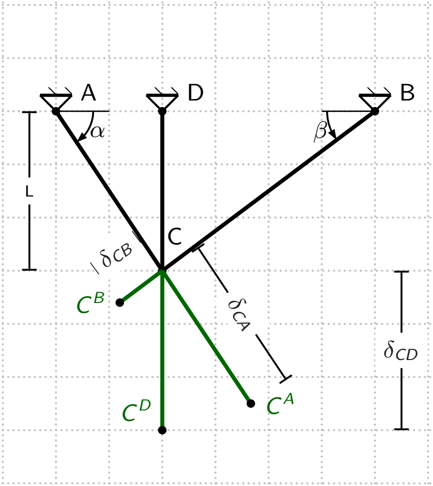
Deformation in each rod:

- rod CA: $\delta_{CA} = \dfrac{L}{\sin \alpha} \dfrac{F_{CA}}{EA}$
- rod CD: $\delta_{CD} = \dfrac{L F_{CD}}{EA}$
- rod CB: $\delta_{CB} = \dfrac{L}{\sin \beta} \dfrac{F_{CB}}{EA}$



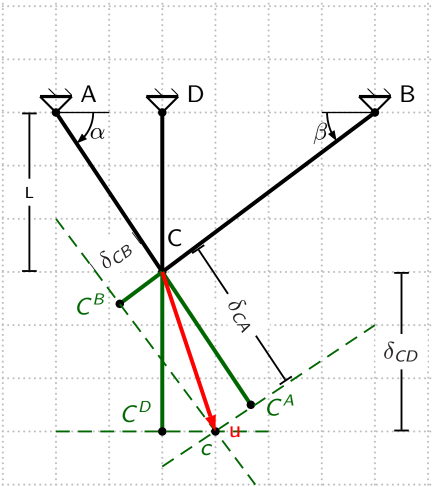
Constitutive relation: for simplicity, we will assume:

- Same Young's modulus, E
- Same cross section, A
- $\delta_{XY} = \dfrac{L_{XY} F_{XY}}{EA}$

Deformation in each rod:

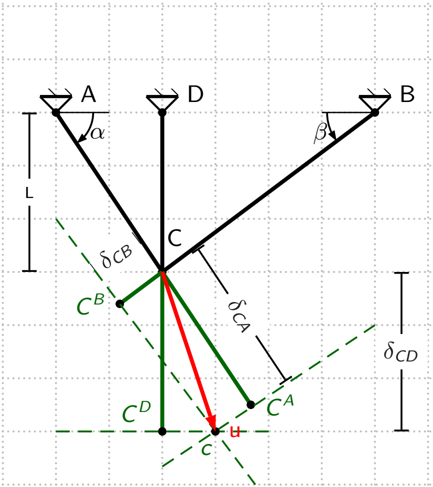
- rod CA: $\delta_{CA} = \dfrac{L}{\sin\alpha} \dfrac{F_{CA}}{EA}$
- rod CD: $\delta_{CD} = \dfrac{L F_{CD}}{EA}$
- rod CB: $\delta_{CB} = \dfrac{L}{\sin\beta} \dfrac{F_{CB}}{EA}$

This adds three equations but also three unknowns, we need to enforce compatibility to close the system of equations



Compatibility condition:

- Small displacements & rotations
- $\delta_{CY} = u_1^C \cos\theta + u_2^C \sin\theta$



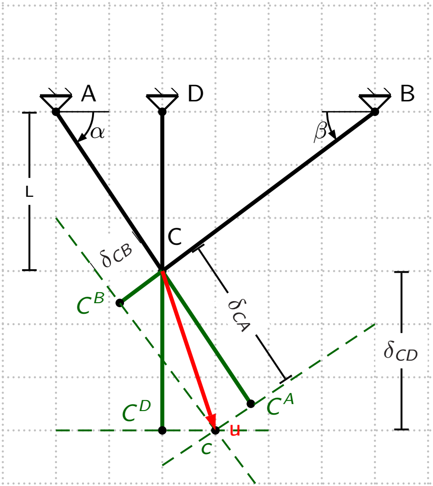
Compatibility condition:

- Small displacements & rotations
- $\delta_{CY} = u_1^C \cos\theta + u_2^C \sin\theta$

Deformation in each rod:

- rod CD: $u_2^C = \delta_{CD}$
- Then:

$$\begin{cases} \delta_{CA} = u_1^c \cos\alpha + \delta_{CD} \sin\alpha \\ \delta_{CB} = -u_1^c \cos\beta + \delta_{CD} \sin\beta \\ \delta_{CD} = u_2^c \end{cases}$$



Compatibility condition:

- Small displacements & rotations
- $\delta_{CY} = u_1^C \cos\theta + u_2^C \sin\theta$

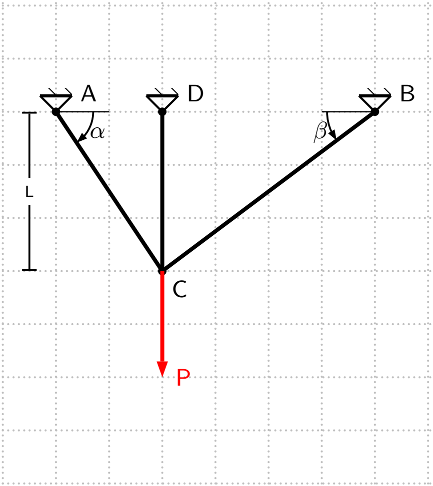
Deformation in each rod:

- rod CD: $u_2^C = \delta_{CD}$
- Then:

$$\begin{cases} \delta_{CA} = u_1^c \cos\alpha + \delta_{CD} \sin\alpha \\ \delta_{CB} = -u_1^c \cos\beta + \delta_{CD} \sin\beta \\ \delta_{CD} = u_2^c \end{cases}$$

Let's put everything together and count unknowns and equations

Putting everything together



Equilibrium

$$\begin{cases} \cos\beta F_{CB} - \cos\alpha F_{CA} = 0 \\ \sin\beta F_{CB} + \sin\alpha F_{CA} + F_{CD} = P \end{cases}$$

Constitutive relation

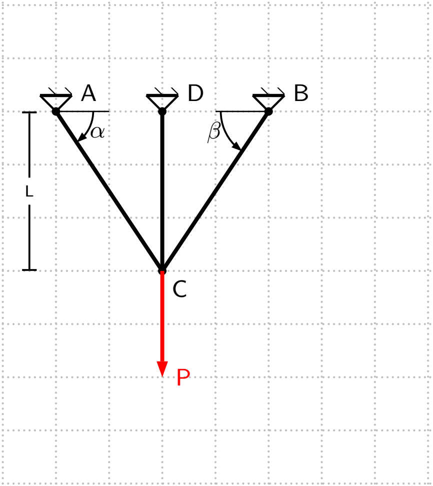
$$\begin{cases} \frac{EA}{L}\sin\alpha\delta_{CA} = F_{CA} \\ \frac{EA}{L}\sin\beta\delta_{CB} = F_{CB} \\ \frac{EA}{L}\delta_{CD} = F_{CD} \end{cases}$$

Compatibility

$$\begin{cases} \delta_{CA} = u_1^c \cos\alpha + u_2^C \sin\alpha \\ \delta_{CB} = u_1^c \cos\beta + u_2^C \sin\beta \\ \delta_{CD} = u_2^C \end{cases}$$

We have 7 unknowns and 7 equations. Let's solve them for $\beta = \alpha$.

Special case: $\beta = \alpha$.



Equilibrium

$$\begin{cases} F_{CB} = F_{CA} \\ 2\sin\alpha F_{CA} + F_{CD} = P \end{cases} \tag{1}$$

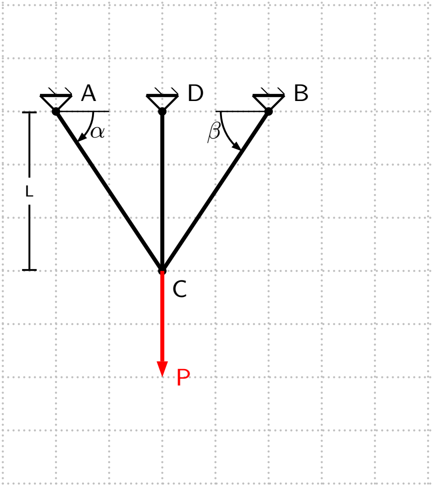
Constitutive relation

$$\begin{cases} \frac{EA}{L}\sin\alpha\delta_{CA} = F_{CA} \\ \frac{EA}{L}\sin\alpha\delta_{CB} = F_{CB} \\ \frac{EA}{L}\delta_{CD} = F_{CD} \end{cases} \tag{2}$$

Compatibility

$$\begin{cases} u_1^C = 0,\ \delta_{CD} = u_2^C \\ \delta_{CA} = \delta_{CB} = u_2^C \sin\alpha \end{cases} \tag{3}$$

Special case: $\beta = \alpha$.



Replacing (3) in (2) and then in (1)

$$\underbrace{2\,\frac{EA}{L}\sin^2\alpha\, u_2^C}_{F_{CA}}\sin\alpha + \underbrace{\frac{EA}{L}u_2^C}_{F_{CD}} = P$$

which gives:

$$\boxed{u_2^C = \frac{P}{EA}L\frac{1}{2\sin^3\alpha + 1}}$$

$$\boxed{F_{CD} = P\frac{1}{2\sin^3\alpha + 1}}$$

$$\boxed{F_{CA} = F_{CB} = P\frac{\sin\alpha}{2\sin^3\alpha + 1}}$$