16.001 Unified Engineering
Materials and Structures

Concept and uses of stress at a point

Reading assignments: CDL: 4.2-4.7

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Outline

Indicial notation and summation convention

A convenient way to write complicated expressions involving vectors and tensor.

Definitions

Free index: A subscript index $i = 1, 3$, $()_i$ will be denoted a *free index* if it is not repeated in the same additive term where the index appears. *Free* means that the index represents **all** the values in its range.

- Latin indices will range from 1 to, $(i, j, k, ... = 1, 2, 3)$,
- greek indices will range from 1 to 2, $(\alpha, \beta, \gamma, ... = 1, 2)$.

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Examples

1. a_{i1} implies a_{11}, a_{21}, a_{31}. (one free index)
2. $x_\alpha y_\beta$ implies $x_1y_1, x_1y_2, x_2y_1, x_2y_2$ (two free indices).
3. a_{ij} implies $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ (two free indices implies 9 values).
4. $\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0$ has a free index (i), therefore it represents three equations:

$$\frac{\partial \sigma_{1j}}{\partial x_j} + b_1 = 0, \ \frac{\partial \sigma_{2j}}{\partial x_j} + b_2 = 0, \ \frac{\partial \sigma_{3j}}{\partial x_j} + b_3 = 0$$

Indicial notation and summation convention, continued

Definitions

Summation convention: When a *repeated index* is found in an expression (inside an additive term) the summation of the terms ranging all the possible values of the indices is implied, i.e.:

$$a_i b_i = \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Note that the choice of index is immaterial:

$$a_i b_i = a_k b_k$$

Examples

1. $a_{ii} = a_{11} + a_{22} + a_{33}$
2. $t_i = \sigma_{ij} n_j$ implies the **three** equations (why?):

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$
$$t_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$$
$$t_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$$

Indicial notation and summation convention, continued

Other important rules about indicial notation:

1. An index cannot appear more than twice in a single additive term, it's either free or repeated only once.

$$a_i = b_{ij} c_j d_j \text{ is INCORRECT}$$

2. In an equation the *lhs* and *rhs*, as well as all the terms on both sides must have the same free indices
   - $a_i b_k = c_{ij} d_{kj}$ free indices i, k, CORRECT
   - $a_i b_k = c_{ij} d_{kj} + e_i f_{jj} + g_k p_i q_r$ INCORRECT, second term is missing free index k and third term has extra free index r

Definition of a basis in $\mathbb{R}^3$

A *basis* in $\mathbb{R}^3$ is given by any set of linearly independent vectors $\mathbf{e}_i$, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. From now on, we will assume that these basis vectors are orthonormal, i.e., they have a unit length and they are orthogonal with respect to each other. This can be expressed using dot products:

$$\mathbf{e}_1.\mathbf{e}_1 = 1, \mathbf{e}_2.\mathbf{e}_2 = 1, \mathbf{e}_3.\mathbf{e}_3 = 1,$$

$$\mathbf{e}_1.\mathbf{e}_2 = 0, \mathbf{e}_1.\mathbf{e}_3 = 0, \mathbf{e}_2.\mathbf{e}_3 = 0, ...$$

The Kronecker Delta

Using indicial notation we can write these expressions in very succinct form as follows:

$$\mathbf{e}_i.\mathbf{e}_j = \delta_{ij}$$

In the last expression the symbol δ_{ij} is defined as the *Kronecker delta*:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

Examples

$$\begin{aligned} a_i\delta_{ij} =& a_1\delta_{11} + a_2\delta_{21} + a_3\delta_{31}, \\ & a_1\delta_{12} + a_2\delta_{22} + a_3\delta_{32}, \\ & a_1\delta_{13} + a_2\delta_{23} + a_3\delta_{33} \\ =& a_1 1 + a_2 0 + a_3 0, \\ & a_1 0 + a_2 1 + a_3 0, \\ & a_1 0 + a_2 0 + a_3 \\ =& a_1, \\ & a_2, \\ & a_3 \end{aligned}$$

or more succinctly: $\boxed{a_i\delta_{ij} = a_j}$, i.e., the Kronecker delta can be thought of an "index replacer".

Vectors IV

Definition of a vector

A **vector v** will be represented as:

$$\mathbf{v} = v_i \mathbf{e}_i = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3$$

The v_i are the *components* of **v** in the basis $\mathbf{e}_i$. These components are the projections of the vector on the basis vectors:

$$\mathbf{v} = v_j \mathbf{e}_j$$

Taking the dot product with basis vector $\mathbf{e}_i$:

$$\mathbf{v}.\mathbf{e}_i = v_j(\mathbf{e}_j.\mathbf{e}_i) = v_j \delta_{ji} = v_i$$

Given two bases $\mathbf{e}_i, \tilde{\mathbf{e}}_k$ and a vector $\mathbf{v}$ whose components in each of these bases are v_i and $\tilde{v}_k$, respectively, we seek to express the components in basis in terms of the components in the other basis. Since the vector is unique:

$$\mathbf{v} = \tilde{v}_m \tilde{\mathbf{e}}_m = v_n \mathbf{e}_n$$

Taking the dot product with $\tilde{\mathbf{e}}_i$:

$$\mathbf{v}.\tilde{\mathbf{e}}_i = \tilde{v}_m(\tilde{\mathbf{e}}_m.\tilde{\mathbf{e}}_i) = v_n(\mathbf{e}_n.\tilde{\mathbf{e}}_i)$$

But $\tilde{v}_m(\tilde{\mathbf{e}}_m.\tilde{\mathbf{e}}_i) = \tilde{v}_m \delta_{mi} = \tilde{v}_i$ from which we obtain:

$$\boxed{\tilde{v}_i = \mathbf{v}.\tilde{\mathbf{e}}_i = v_j(\mathbf{e}_j.\tilde{\mathbf{e}}_i)}$$

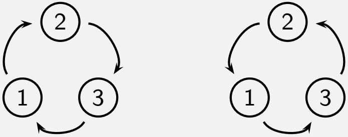
Note that $(\mathbf{e}_j.\tilde{\mathbf{e}}_i)$ are the *direction cosines* of the basis vectors of one basis on the other basis:

$$\mathbf{e}_j.\tilde{\mathbf{e}}_i = \|\mathbf{e}_j\| \|\tilde{\mathbf{e}}_i\| \cos \widehat{\mathbf{e}_j \tilde{\mathbf{e}}_i}$$

Definition

Permutation Tensor:

$$\epsilon_{mnp} = \begin{cases} 0 & \text{when any two indices are equal} \\ 1 & \text{when } mnp \text{ is an even permutation of } 1,2,3 \\ -1 & \text{when } mnp \text{ is an odd permutation of } 1,2,3 \end{cases}$$



Even permutation   Odd permutation

Examples

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1,\ \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1,\ \epsilon_{112} = \epsilon_{233} = \epsilon_{222} = \cdots = 0,$$

Source and usefulness: Cross products of cartesian basis vectors

$$\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{0}, \mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{0}, \mathbf{e}_3 \times \mathbf{e}_3 = \mathbf{0},$$

$$\mathbf{e}_1 \times \mathbf{e}_2 = (+1)\mathbf{e}_3, \mathbf{e}_2 \times \mathbf{e}_3 = (+1)\mathbf{e}_1, \mathbf{e}_3 \times \mathbf{e}_1 = (+1)\mathbf{e}_2,$$

$$\mathbf{e}_1 \times \mathbf{e}_3 = (-1)\mathbf{e}_2, \mathbf{e}_3 \times \mathbf{e}_2 = (-1)\mathbf{e}_1, \mathbf{e}_2 \times \mathbf{e}_1 = (-1)\mathbf{e}_3,$$

These can all be captured by

$$\boxed{\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k}$$

Example

$$\mathbf{e}_1 \times \mathbf{e}_2 = \underbrace{\epsilon_{121}}_{=0} \mathbf{e}_1 + \underbrace{\epsilon_{122}}_{=0} \mathbf{e}_2 + \underbrace{\epsilon_{123}}_{=1} \mathbf{e}_3 = \mathbf{e}_3$$

We can also observe that

$$(\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k = (\epsilon_{ijl}\mathbf{e}_l) \cdot \mathbf{e}_k = \epsilon_{ijl}\delta_{kl} = \epsilon_{ijk}$$

The permutation tensor can be used to express some of the other familiar vector operations involving cross products.

Cross product of two vectors

$$\boxed{\mathbf{v} \times \mathbf{w} = (v_i\mathbf{e}_i) \times (w_j)\mathbf{e}_j) = v_i w_j (\mathbf{e}_i \times \mathbf{e}_j) = v_i w_j \epsilon_{ijk} \mathbf{e}_k}$$

Mixed or triple product of three vectors

$$\boxed{(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = v_i w_j \epsilon_{ijl} \mathbf{e}_l \cdot (u_k \mathbf{e}_k) = v_i w_j u_k \epsilon_{ijk}}$$

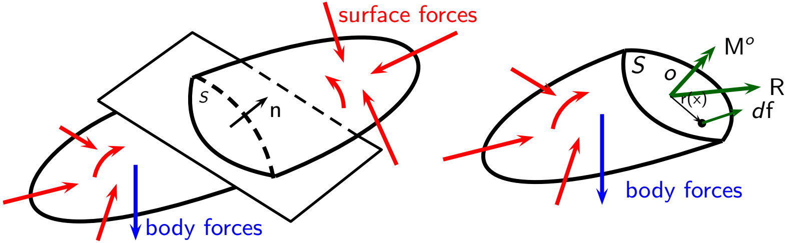
Since the triple product can also be obtained from the determinant of the 3x3 matrix made of the components of the three vectors (either arranged in row or column form), we can use this to express the determinant of a 3x3 matrix A with components a_{ij} as follows. Assign the rows of the matrix to the components of the vectors above as follows: $v_i = a_{1i}, w_j = a_{2j}, u_k = a_{3k}$, then:

Determinant of a 3x3 Matrix

$$\boxed{|A| = a_{1i}a_{2j}a_{3k}\epsilon_{ijk}}$$

Outline

Consider a body in equilibrium under external surface and body forces:



We imagine a cut through the body with a plane defining the surface internal surface S with normal $\mathbf{n}$. The FBD on the right shows the resultant internal force ($\mathbf{R}$) and moment ($\mathbf{M}^o$) required from equilibrium for the left part. We assume that these are provided by the collective action of infinitesimal pointwise forces $d\mathbf{f}$ acting on the points of S.

$\mathbf{R}$ and $\mathbf{M}^o$ are therefore obtained as the following integrals:

$$\mathbf{R} = \int_S d\mathbf{f}$$

$$\mathbf{M}^o = \int_S \mathbf{r} \times d\mathbf{f}$$

It should be clear that $\mathbf{t}$ is a force per unit area defined at each point of surface S obtained as the limiting value of the resultant force $\mathbf{f}$ acting on a (finite) surface area element ΔS when this tends to zero.

Definition

The *stress vector* at a point on ΔS is defined as:

$$\boxed{\mathbf{t} = \lim_{\Delta S \to 0} \frac{\mathbf{f}}{\Delta S}} \tag{1}$$

Notes:

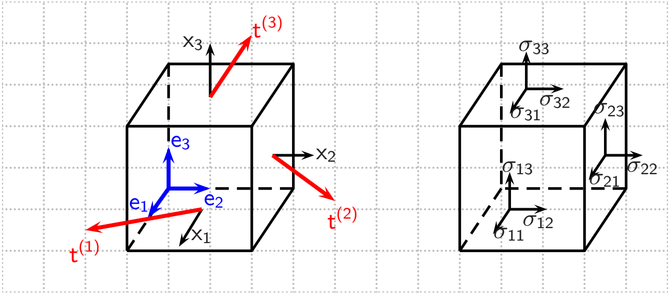
- The integral of the stress vector in the area defines the resultant internal force necessary to keep the left side in equilibrium (as we discussed before)

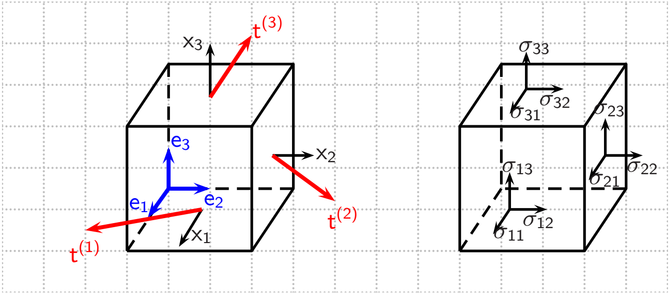
$$\mathbf{R} = \int_S \mathbf{t} dS$$

- Similarly, the resultant internal moment vector $\mathbf{M}^o$ with respect to a point o is given by:

$$\mathbf{M}^o = \int_S \mathbf{r} \times \mathbf{t} dS$$

In other words, the internal force and moment vectors are the equipollent force system corresponding to the colective action of the continuously distributed stress vectors **t** on the cut surface.
If the cut had gone through the same point under consideration but along a plane with a different normal, the stress vector would have been different. Let's consider the three stress vectors $\mathbf{t}^{(i)}$ acting on the planes normal to the coordinate axes.



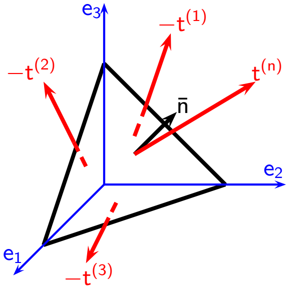


Let's also decompose each $\mathbf{t}^{(i)}$ in its three components in the coordinate system $\mathbf{e}_i$ (this can be done for any vector) as (see Figure):

$$\mathbf{t}^{(i)} = \sigma_{ij}\mathbf{e}_j \tag{2}$$

σ_{ij} is the component of the stress vector $\mathbf{t}^{(i)}$ along the $\mathbf{e}_j$ direction.

Different planes passing through the point with different normals will have different stress vectors $\mathbf{t}^{(\mathbf{n})}$. Is there a relation among them? To answer this invoke equilibrium of the (shrinking) tetrahedron of material:



Cauchy's tetrahedron: equilibrium of a tetrahedron shrinking to a point.

The area of the faces of the tetrahedron are ΔS_1, ΔS_2, ΔS_3 and ΔS.
We have used Newton's third law of action and reaction: $\mathbf{t}^{(-\mathbf{n})} = -\mathbf{t}^{(\mathbf{n})}$. To enforce equilibrium, we must consider the force vectors (stress vectors multiplied by respective areas) acting on each face of the tetrahedron:

$$\mathbf{t}^{(\mathbf{n})}\Delta S - \mathbf{t}^{(1)}\Delta S_1 - \mathbf{t}^{(2)}\Delta S_2 - \mathbf{t}^{(3)}\Delta S_3 = 0$$

The surface area elements are related by the following formula: $\Delta S n_i = \Delta S_i$. Proof in the following mathematical aside:

By virtue of Green's Theorem:

$$\int_V \nabla \phi dV = \int_S \mathbf{n} \phi dS$$

applied to the function $\phi = 1$, we get

$$0 = \int_S \mathbf{n} dS$$

which applied to our tetrahedron gives:

$$0 = \Delta S \mathbf{n} - \Delta S_1 \mathbf{e}_1 - \Delta S_2 \mathbf{e}_2 - \Delta S_3 \mathbf{e}_3$$

If we take the scalar product of this equation with $\mathbf{e}_i$, we obtain:

$$\Delta S(\mathbf{n} \cdot \mathbf{e}_i) = \Delta S_i$$

or

$$\boxed{\Delta S_i = \Delta S n_i}$$

Replace in equilibrium expression:

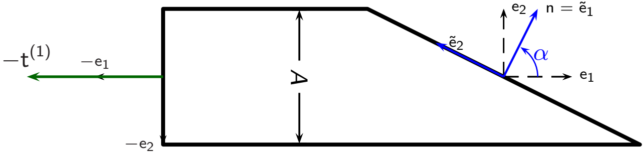
$$\Delta S\big(\mathbf{t}^{(\mathbf{n})} - \underbrace{n_1}_{(\mathbf{n}\cdot\mathbf{e}_1)}\mathbf{t}^{(1)} - \underbrace{n_2}_{(\mathbf{n}\cdot\mathbf{e}_2)}\mathbf{t}^{(2)} - \underbrace{n_3}_{(\mathbf{n}\cdot\mathbf{e}_3)}\mathbf{t}^{(3)}\big) = 0$$

which can be written more simply (using summation convention) as:

$$\mathbf{t}^{(\mathbf{n})} = \underbrace{(\mathbf{n}\cdot\mathbf{e}_i)}_{n_i}\mathbf{t}^{(i)} \tag{3}$$

Example

Consider a cut at an angle α in a truss member of cross sectional area A and subject to a force of magnitude F. The bar is subject to a uniform uniaxial stress $\sigma = \frac{F}{A}$. The stress vector at any point on a plane of normal $\mathbf{e}_1$ is (see figure) $\mathbf{t}^{(1)} = \sigma \mathbf{e}_1$. $\mathbf{t}^{(2)} = \mathbf{t}^{(3)} = \mathbf{0}$. What is $\mathbf{t}^{(\mathbf{n})}$ where $\mathbf{n} = \tilde{\mathbf{e}}_1 = \cos(\alpha)\mathbf{e}_1 + \sin(\alpha)\mathbf{e}_2$?



$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^{(\tilde{\mathbf{e}}_1)} = n_i \mathbf{t}^{(i)} = n_1 \mathbf{t}^{(1)} = \cos(\alpha)\,\sigma \mathbf{e}_1 = \frac{F}{\frac{A}{\cos(\alpha)}} \mathbf{e}_1$$

This gives us $\mathbf{t}^{(\mathbf{n})}$ in the basis $\mathbf{e}_i$. What about in basis $\tilde{\mathbf{e}}_i$?

$$\begin{aligned}\mathbf{t}^{(\tilde{\mathbf{e}}_1)} &= \left(\mathbf{t}^{(\tilde{\mathbf{e}}_1)} \cdot \tilde{\mathbf{e}}_1\right) \tilde{\mathbf{e}}_1 + \left(\mathbf{t}^{(\tilde{\mathbf{e}}_2)} \cdot \tilde{\mathbf{e}}_2\right) \tilde{\mathbf{e}}_2 = \left(\mathbf{t}^{(\tilde{\mathbf{e}}_1)} \cdot \tilde{\mathbf{e}}_i\right) \tilde{\mathbf{e}}_i \\ &= (\cos(\alpha)\,\sigma \mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\cos(\alpha)\,\sigma \mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2 \\ &= \sigma \cos^2(\alpha)\tilde{\mathbf{e}}_1 + \sigma \cos(\alpha)(-\sin(\alpha))\tilde{\mathbf{e}}_2\end{aligned}$$

Going back to Eqn. (3), we can also pull $\mathbf{n}$ as a "common factor" and create a new type of mathematical expression (tensor product):

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \left(\mathbf{e}_1 \mathbf{t}^{(1)} + \mathbf{e}_2 \mathbf{t}^{(2)} + \mathbf{e}_3 \mathbf{t}^{(3)}\right) = \mathbf{n} \cdot \left(\mathbf{e}_1 \otimes \mathbf{t}^{(1)} + \mathbf{e}_2 \otimes \mathbf{t}^{(2)} + \mathbf{e}_3 \otimes \mathbf{t}^{(3)}\right) \quad (4)$$

The factor in parenthesis is the definition of the *Cauchy stress tensor* $\boldsymbol{\sigma}$:

Definition

$$\boxed{\boldsymbol{\sigma} = \mathbf{e}_1 \otimes \mathbf{t}^{(1)} + \mathbf{e}_2 \otimes \mathbf{t}^{(2)} + \mathbf{e}_3 \otimes \mathbf{t}^{(3)} = \mathbf{e}_i \otimes \mathbf{t}^{(i)}}$$
$$\boxed{\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma}} \quad (5)$$

Note these are tensorial expressions (independent of the vector and tensor components in a particular coordinate system). To obtain the tensorial components in our rectangular system we replace the expressions of $\mathbf{t}^{(i)}$ from Eqn.(2)

Definition (Stress tensor representation in cartesian coordinate basis $\mathbf{e}_i$)

$$\boldsymbol{\sigma} = \mathbf{e}_i \underbrace{\sigma_{ij}\mathbf{e}_j}_{\mathbf{t}^{(i)}} = \sigma_{ij}\mathbf{e}_i\mathbf{e}_j \tag{6}$$

where

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

are the components of the stress tensor $\boldsymbol{\sigma}$ in the cartesian coordinate system $\mathbf{e}_i$. Note that σ_{ij} represent the cartesian components of the stress vectors acting on the planes with normals $\mathbf{e}_i$, i.e. $\mathbf{t}^{(i)}$

Definition of stress tensor III

The cartesian components of the stress vector on the plane with normal **n** can be obtained by noticing that:

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \sigma_{ij}\mathbf{e}_i\mathbf{e}_j = \sigma_{ij}\left(\mathbf{n} \cdot \mathbf{e}_i\right)\mathbf{e}_j = \left(\sigma_{ij} n_i\right)\mathbf{e}_j \tag{7}$$

$$t_j^{(\mathbf{n})} = t_j(\mathbf{n}) = \sigma_{ij} n_i \tag{8}$$