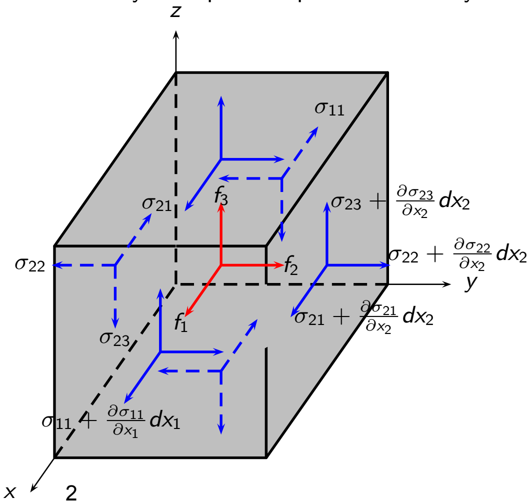
Outline

Stress equilibrium at a point I

In general, the state of stress in general varies throughout a solid body, i.e. the stress tensor is a field (a function in 3D space) $\sigma_{ij} = \sigma_{ij}(\mathrm{x})$ Equilibrium imposes restrictions on the way the stress field can vary from point to point in the body.

To see this, consider opposite faces of a differential volume element subject to a varying stress state and body force field $f_i(\mathrm{x})$. To first order (in a Taylor expansion of the stress components), the volume element is subject to the stress state shown in the figure (some of the labels have been omitted for clarity).



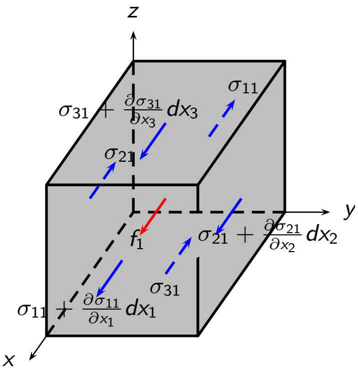
Stress equilibrium at a point: requirements

$$\sum F_1 : \; 0 = f_1 dx_1 dx_2 dx_3 +$$
$$\left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\right) dx_2 dx_3 - \sigma_{11} dx_2 dx_3 +$$
$$\left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) dx_3 dx_1 - \sigma_{21} dx_3 dx_1 +$$
$$\left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3\right) dx_1 dx_2 - \sigma_{31} dx_1 dx_2$$

(1)

(2)

(3)

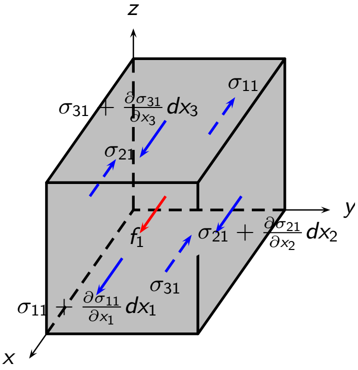


$$\sum F_1 : \ 0 = f_1 dx_1 dx_2 dx_3 +$$
$$\left(\cancel{\sigma_{11}} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\right) dx_2 dx_3 - \cancel{\sigma_{11}} dx_2 dx_3 +$$
$$\left(\cancel{\sigma_{21}} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) dx_3 dx_1 - \cancel{\sigma_{21}} dx_3 dx_1 +$$
$$\left(\cancel{\sigma_{31}} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3\right) dx_1 dx_2 - \cancel{\sigma_{31}} dx_1 dx_2$$

(1)

(2)

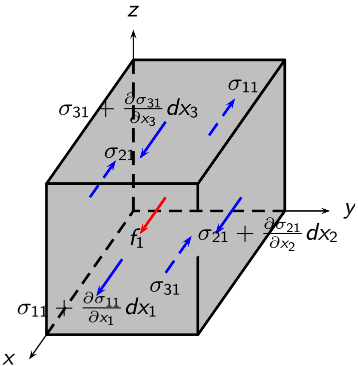
(3)



$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = 0 \qquad (1)$$

(2)

(3)

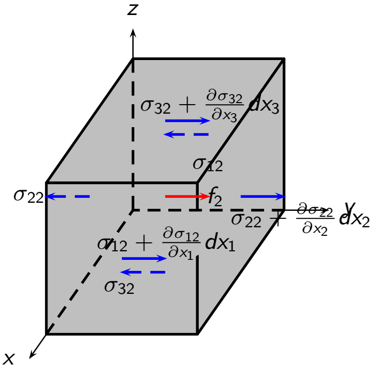


$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = 0 \qquad (1)$$

$$\sum F_2 :\ 0 = f_2 dx_1 dx_2 dx_3 +$$
$$\left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1} dx_1\right) dx_2 dx_3 - \sigma_{12} dx_2 dx_3 +$$
$$\left(\sigma_{22} + \frac{\partial \sigma_{22}}{\partial x_2} dx_2\right) dx_3 dx_1 - \sigma_{22} dx_3 dx_1 +$$
$$\left(\sigma_{32} + \frac{\partial \sigma_{32}}{\partial x_3} dx_3\right) dx_1 dx_2 - \sigma_{32} dx_1 dx_2$$

(2)

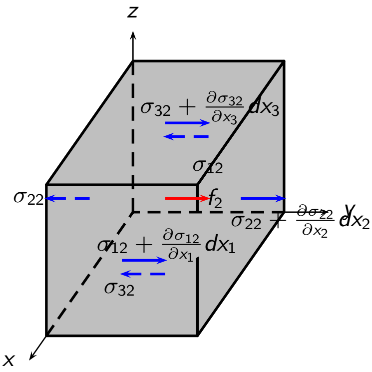
(3)



$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3} + f_1 = 0 \qquad (1)$$

$$\sum F_2 : \; 0 = f_2 dx_1 dx_2 dx_3 +$$
$$\left(\cancel{\sigma_{12}} + \frac{\partial\sigma_{12}}{\partial x_1} dx_1\right) dx_2 dx_3 - \cancel{\sigma_{12}} dx_2 dx_3 +$$
$$\left(\cancel{\sigma_{22}} + \frac{\partial\sigma_{22}}{\partial x_2} dx_2\right) dx_3 dx_1 - \cancel{\sigma_{22}} dx_3 dx_1 +$$
$$\left(\cancel{\sigma_{32}} + \frac{\partial\sigma_{32}}{\partial x_3} dx_3\right) dx_1 dx_2 - \cancel{\sigma_{32}} dx_1 dx_2 \qquad (2)$$

$$(3)$$

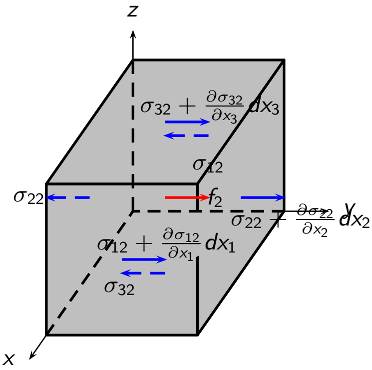


Stress equilibrium at a point: requirements

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = 0 \quad (1)$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = 0 \quad (2)$$

$$(3)$$

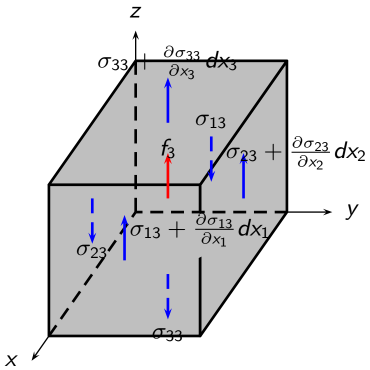


Stress equilibrium at a point: requirements

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = 0 \quad (1)$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = 0 \quad (2)$$

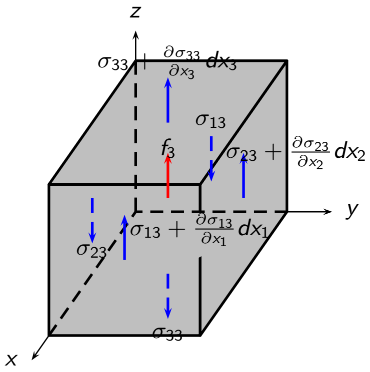
$$\begin{aligned} \sum F_3 :\ 0 = f_3 dx_1 dx_2 dx_3 + \\ \left(\sigma_{13} + \frac{\partial \sigma_{13}}{\partial x_1} dx_1\right) dx_2 dx_3 - \sigma_{13} dx_2 dx_3 + \\ \left(\sigma_{23} + \frac{\partial \sigma_{23}}{\partial x_2} dx_2\right) dx_3 dx_1 - \sigma_{23} dx_3 dx_1 + \\ \left(\sigma_{33} + \frac{\partial \sigma_{33}}{\partial x_3} dx_3\right) dx_1 dx_2 - \sigma_{33} dx_1 dx_2 \end{aligned} \quad (3)$$



$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = 0 \qquad (1)$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = 0 \qquad (2)$$

$$\begin{aligned} \sum F_3 :\ 0 = f_2 dx_1 dx_2 dx_3 + \\ \left(\cancel{\sigma_{13}} + \frac{\partial \sigma_{13}}{\partial x_1} dx_1\right) dx_2 dx_3 - \cancel{\sigma_{13}} dx_2 dx_3 + \\ \left(\cancel{\sigma_{23}} + \frac{\partial \sigma_{23}}{\partial x_2} dx_2\right) dx_3 dx_1 - \cancel{\sigma_{23}} dx_3 dx_1 + \\ \left(\cancel{\sigma_{33}} + \frac{\partial \sigma_{33}}{\partial x_3} dx_3\right) dx_1 dx_2 - \cancel{\sigma_{33}} dx_1 dx_2 \end{aligned} \qquad (3)$$

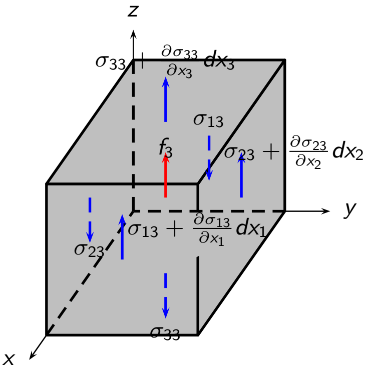


Stress equilibrium at a point: requirements

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3} + f_1 = 0 \qquad (1)$$

$$\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{32}}{\partial x_3} + f_2 = 0 \qquad (2)$$

$$\frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} + f_2 = 0 \qquad (3)$$



Differential equations of stress equilibrium

$$\boxed{\frac{\partial \sigma_{ji}}{\partial x_j} + f_i = 0}$$

In tensor form:

$$\boxed{\nabla \cdot \sigma + \mathrm{f} = 0}$$

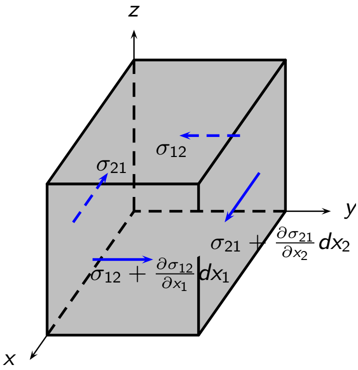
- must be satisfied by the stress field at every point in a stressed body or structure
- constitutes a system of three partial differential equations relating the 6 independent stress components

Specialization to 1D and 2D stress states:

Derive the differential equations of stress equilibrium for 1D and 2D stress states.

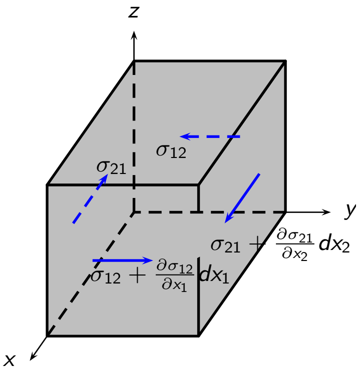
Equilibrium of moments with respect to cube center:

$$\sum M_3 = 0 : \; 0 = \left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1} dx_1\right) dx_2 dx_3 \frac{dx_1}{2} + \sigma_{12} dx_2 dx_3 \frac{dx_1}{2} - \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) dx_1 dx_3 \frac{dx_2}{2} - \sigma_{21} dx_1 dx_3 \frac{dx_2}{2} \tag{4}$$



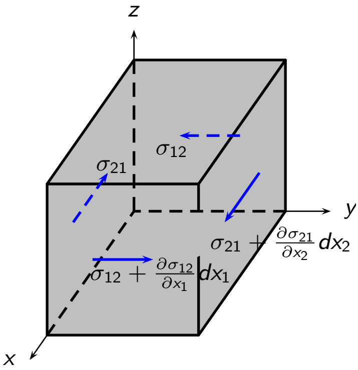
Equilibrium of moments with respect to cube center:

$$\sum M_3 = 0 : \; 0 = \left(\sigma_{12} + \underbrace{\frac{\partial \sigma_{12}}{\partial x_1} dx_1}_{h.o.t}\right) dx_2 dx_3 \frac{dx_1}{2} + \sigma_{12} dx_2 dx_3 \frac{dx_1}{2} - \left(\sigma_{21} + \underbrace{\frac{\partial \sigma_{21}}{\partial x_2} dx_2}_{h.o.t}\right) dx_1 dx_3 \frac{dx_2}{2} - \sigma_{21} dx_1 dx_3 \frac{dx_2}{2} \quad (4)$$



Equilibrium of moments with respect to cube center:

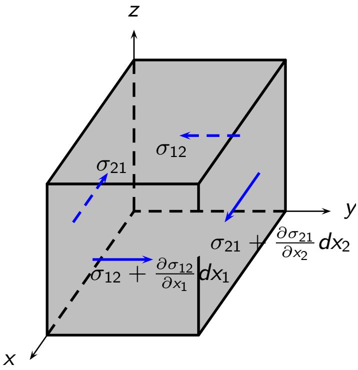
$$\begin{aligned} \sum M_3 = 0 :\ 0 = \\ dx_2 dx_3 \frac{dx_1}{2}(\sigma_{12} + \sigma_{12}) \\ -dx_1 dx_3 \frac{dx_2}{2}(\sigma_{21} + \sigma_{21}) \\ \sigma_{12} - \sigma_{21} = 0 \\ \boxed{\sigma_{12} = \sigma_{21}} \end{aligned} \tag{4}$$



$\sum M_1 = 0$ gives: $\sigma_{23} = \sigma_{32}$, and $\sum M_2 = 0$ gives $\sigma_{13} = \sigma_{31}$. In summary:

Symmetry of stress tensor

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$



$$\nabla \cdot \sigma = \left(\frac{\partial()}{\partial r}\mathrm{e}_r + \frac{1}{r}\frac{\partial()}{\partial \theta}\mathrm{e}_\theta\right) \cdot \left(\tilde{\sigma}_{mn}\tilde{\mathrm{e}}_m \otimes \tilde{\mathrm{e}}_n\right)$$

$$= \left(\frac{\partial()}{\partial r}\mathrm{e}_r + \frac{1}{r}\frac{\partial()}{\partial \theta}\mathrm{e}_\theta\right) \cdot \left(\sigma_{rr}\mathrm{e}_r \otimes \mathrm{e}_r + \sigma_{r\theta}\mathrm{e}_r \otimes \mathrm{e}_\theta + \sigma_{\theta r}\mathrm{e}_\theta \otimes \mathrm{e}_r + \sigma_{\theta\theta}\mathrm{e}_\theta \otimes \mathrm{e}_\theta\right)$$

$$= \frac{\partial(\sigma_{rr}\mathrm{e}_r \otimes \mathrm{e}_r + \sigma_{r\theta}\mathrm{e}_r \otimes \mathrm{e}_\theta + \sigma_{\theta r}\mathrm{e}_\theta \otimes \mathrm{e}_r + \sigma_{\theta\theta}\mathrm{e}_\theta \otimes \mathrm{e}_\theta)}{\partial r} \cdot \mathrm{e}_r$$

$$+ \frac{1}{r}\frac{\partial(\sigma_{rr}\mathrm{e}_r \otimes \mathrm{e}_r + \sigma_{r\theta}\mathrm{e}_r \otimes \mathrm{e}_\theta + \sigma_{\theta r}\mathrm{e}_\theta \otimes \mathrm{e}_r + \sigma_{\theta\theta}\mathrm{e}_\theta \otimes \mathrm{e}_\theta)}{\partial \theta} \cdot \mathrm{e}_\theta$$

$$= \left(\frac{\partial \sigma_{rr}}{\partial r}\mathrm{e}_r \otimes \mathrm{e}_r + \frac{\partial \sigma_{r\theta}}{\partial r}\mathrm{e}_r \otimes \mathrm{e}_\theta + \frac{\partial \sigma_{\theta r}}{\partial r}\mathrm{e}_\theta \otimes \mathrm{e}_r + \frac{\partial \sigma_{\theta\theta}}{\partial r}\mathrm{e}_\theta \otimes \mathrm{e}_\theta\right) \cdot \mathrm{e}_r + \Bigg\{$$

$$\frac{\partial \sigma_{rr}}{\partial \theta}\mathrm{e}_r \otimes \mathrm{e}_r + \sigma_{rr}\left(\overbrace{\frac{\partial \mathrm{e}_r}{\partial \theta}}^{\mathrm{e}_\theta} \otimes \mathrm{e}_r + \mathrm{e}_r \otimes \overbrace{\frac{\partial \mathrm{e}_r}{\partial \theta}}^{\mathrm{e}_\theta}\right) + \frac{\partial \sigma_{r\theta}}{\partial \theta}\mathrm{e}_r \otimes \mathrm{e}_\theta + \sigma_{r\theta}\left(\overbrace{\frac{\partial \mathrm{e}_r}{\partial \theta}}^{\mathrm{e}_\theta} \otimes \mathrm{e}_\theta + \mathrm{e}_r \otimes \overbrace{\frac{\partial \mathrm{e}_\theta}{\partial \theta}}^{-\mathrm{e}_r}\right) +$$

In Cylindrical coordinates II

$$\frac{\partial \sigma_{\theta r}}{\partial \theta}\mathrm{e}_\theta \otimes \mathrm{e}_r + \sigma_{\theta r}\left(\overbrace{\frac{\partial \mathrm{e}_\theta}{\partial \theta}}^{-\mathrm{e}_r}\otimes \mathrm{e}_r + \mathrm{e}_\theta \otimes \overbrace{\frac{\partial \mathrm{e}_r}{\partial \theta}}^{\mathrm{e}_\theta}\right) + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\mathrm{e}_\theta \otimes \mathrm{e}_\theta + \sigma_{\theta\theta}\left(\overbrace{\frac{\partial \mathrm{e}_\theta}{\partial \theta}}^{-\mathrm{e}_r}\otimes \mathrm{e}_\theta + \mathrm{e}_\theta \otimes \overbrace{\frac{\partial \mathrm{e}_\theta}{\partial \theta}}^{-\mathrm{e}_r}\right)$$

$$\Big\} \cdot \frac{1}{r}\mathrm{e}_\theta$$

$$= \frac{\partial \sigma_{rr}}{\partial r}\mathrm{e}_r \times 1 + \cancel{\frac{\partial \sigma_{r\theta}}{\partial r}\mathrm{e}_r \times 0} + \frac{\partial \sigma_{\theta r}}{\partial r}\mathrm{e}_\theta \times 1 + \cancel{\frac{\partial \sigma_{\theta\theta}}{\partial r}\mathrm{e}_\theta \times 0} + \frac{1}{r}\Big\{$$

$$\cancel{\frac{\partial \sigma_{rr}}{\partial \theta}\mathrm{e}_r \times 0} + \sigma_{rr}\left(\cancel{\mathrm{e}_\theta \times 0} + \mathrm{e}_r \times 1\right) + \frac{\partial \sigma_{r\theta}}{\partial \theta}\mathrm{e}_r \times 1 + \sigma_{r\theta}\left(\mathrm{e}_\theta \times 1 + \cancel{\mathrm{e}_r \times (-0)}\right) +$$

$$\cancel{\frac{\partial \sigma_{\theta r}}{\partial \theta}\mathrm{e}_\theta \times 0} + \sigma_{\theta r}\left(\cancel{-\mathrm{e}_r \times 0} + \mathrm{e}_\theta \times 1\right) + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\mathrm{e}_\theta \times 1 + \sigma_{\theta\theta}\left(-\mathrm{e}_r \times 1 + \cancel{\mathrm{e}_\theta \times (-0)}\right)\Big\}$$

$$= \left\{\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\left(\sigma_{rr} + \frac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta}\right)\right\}\mathrm{e}_r + \left\{\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\left(\sigma_{r\theta} + \sigma_{\theta r} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right)\right\}\mathrm{e}_\theta$$

From the components of the divergence of the stress tensor in cylindrical coordinates, we can obtain the equations of equilibrium:

$$\boxed{\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0} \tag{4}$$

$$\boxed{\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + 2\frac{\sigma_{r\theta}}{r} + f_\theta = 0} \tag{5}$$

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