16.001 Unified Engineering
Materials and Structures

Deformation and Strain

Reading assignments: CDL 4.8-4.15

Instructors: Raúl Radovitzky, Zachary Cordero

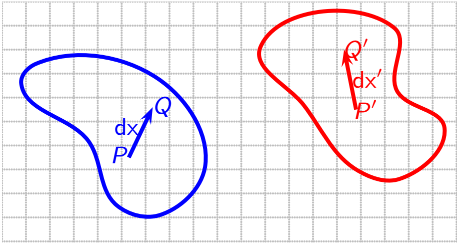
Teaching Assistants: Grégoire Chomette, Michelle Xu, and Daniel Pickard

Massachusetts Institute of Technology
Department of Aeronautics & Astronautics

Outline

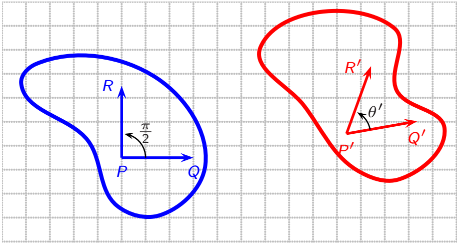
As we discussed at the beginning of the course, materials can resist loads by virtue of their ability to deform. Some times material deformations are imperceptible with the naked eye but can be captured by specialized equipment. Deformations can also occur due to temperature. In summary, we refer to deformation as the changes in size and shape of the body.

Deformation and the Strain Tensor II



Normal strain: is the elongation or contraction of a line segment per unit length. For a line segment **dx** of undeformed infinitesimal length $\|\mathbf{dx}\| = \Delta S$, and deformed length $\Delta S' = \|\mathbf{dx}'\|$, we define the normal strain in the direction of the undeformed segment as:

$$\varepsilon = \lim_{\Delta S \to 0} \frac{\Delta S' - \Delta S}{\Delta S}$$



Engineering shear strain: γ is the change in angle between two perpedicular line segments in the undeformed configuration.

$$\gamma = \frac{\pi}{2} - \theta'$$

A note on units: strains both normal and shear are dimensionless quantities

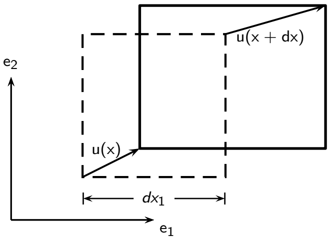
Strain - displacement relations: Deformation described by displacement field $\mathbf{u}(\mathbf{x}) = u_i(\mathbf{x})\mathbf{e}_i$. Small displacement gradients assumed throughout.

Extensional strains: Measure elongation of volume element in $\mathbf{x}$:

$$\varepsilon_{11} = \frac{\Delta L}{L}$$

$$= \frac{\left(\cancel{u_1} + \frac{\partial u_1}{\partial x_1} dx_1\right) - \cancel{u_1}}{dx_1}$$

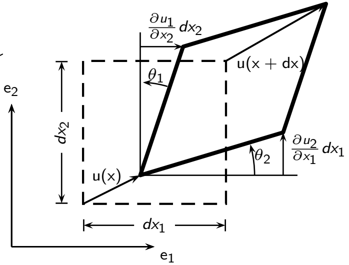
$$\boxed{\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}}$$



Shear strains: Measure changes of angle in volume element: Initial angle: $\frac{\pi}{2}$. Deformed angle: $\frac{\pi}{2} - (\theta_1 + \theta_2)$.
Engineering shear strain (Total angle change): $\gamma_{12} = \theta_1 + \theta_2$

$$\theta_1 \sim \tan\theta_1 = \frac{\frac{\partial u_1}{\partial x_2}\cancel{dx_2}}{\cancel{dx_2}}, \ \theta_2 \sim \tan\theta_2 = \frac{\frac{\partial u_2}{\partial x_1}\cancel{dx_1}}{\cancel{dx_1}}$$

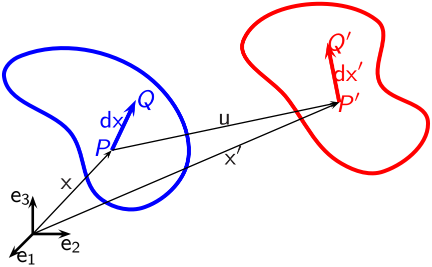
$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$



Tensor strain components defined as half the total angle change: :
$\varepsilon_{12} = \varepsilon_{21} = \frac{\gamma_{12}}{2}$

$$\boxed{\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)}$$

In general, deformations are not uniform throughout the body of a loaded structure. For example, some parts of the body may elongate, others may contract. This will also depend on the orientation. We wish to characterize the local state of deformation at each point of a material body. We therefore describe deformation by looking at the change of length of infinitesimal line segments of arbitrary directions at a point.



Deformation described by *deformation mapping*:

$$\mathbf{x}' = \varphi(\mathbf{x}) = \mathbf{x} + \mathbf{u} \quad (1)$$

We seek to characterize the local state of deformation of the material in a neighborhood of a point P.

Consider two points P and Q in the undeformed:

$$P : \mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i \tag{2}$$

$$Q : \mathbf{x} + \mathbf{dx} = (x_i + dx_i)\mathbf{e}_i \tag{3}$$

and deformed

$$P' : \mathbf{x}' = \varphi_1(\mathbf{x})\mathbf{e}_1 + \varphi_2(\mathbf{x})\mathbf{e}_2 + \varphi_3(\mathbf{x})\mathbf{e}_3 = \varphi_i(\mathbf{x})\mathbf{e}_i \tag{4}$$

$$Q' : \mathbf{x}' + \mathbf{dx}' = \big(\varphi_i(\mathbf{x}) + d\varphi_i\big)\mathbf{e}_i \tag{5}$$

configurations. In this expression,

$$\mathbf{dx}' = d\varphi_i\mathbf{e}_i \tag{6}$$

Expressing the differentials $d\varphi_i$ in terms of the partial derivatives of the functions $\varphi_i(x_j\mathbf{e}_j)$:

$$d\varphi_1 = \frac{\partial\varphi_1}{\partial x_1}dx_1 + \frac{\partial\varphi_1}{\partial x_2}dx_2 + \frac{\partial\varphi_1}{\partial x_3}dx_3, \tag{7}$$

and similarly for $d\varphi_2, d\varphi_3$, in index notation:

$$d\varphi_i = \frac{\partial \varphi_i}{\partial x_j} dx_j \tag{8}$$

Replacing in equation (5):

$$Q' : \mathbf{x}' + \mathbf{dx}' = \left(\varphi_i + \frac{\partial \varphi_i}{\partial x_j} dx_j\right)\mathbf{e}_i \tag{9}$$

$$\mathbf{dx}' = \frac{\partial \varphi_i}{\partial x_j} dx_j \mathbf{e}_i \tag{10}$$

We now compute the change in length of the segment $\overrightarrow{PQ}$ which deformed into segment $\overrightarrow{P'Q'}$. Undeformed length (to the square):

$$ds^2 = \|\mathbf{dx}\|^2 = \mathbf{dx} \cdot \mathbf{dx} = dx_i dx_i \tag{11}$$

Deformed length (to the square):

$$(ds')^2 = \|\mathbf{dx}'\|^2 = \mathbf{dx}' \cdot \mathbf{dx}' = \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k \tag{12}$$

The change in length of segment $\overrightarrow{PQ}$ is then given by the difference between equations (12) and (11):

$$(ds')^2 - ds^2 = \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k - dx_i dx_i \tag{13}$$

We want to extract as common factor the differentials. To this end we observe that:

$$dx_i dx_i = dx_j dx_k \delta_{jk} \tag{14}$$

Then:

$$
\begin{aligned}
(ds')^2 - ds^2 &= \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k - dx_j dx_k \delta_{jk} \\
&= \underbrace{\left(\frac{\partial \varphi_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_k} - \delta_{jk} \right)}_{2\varepsilon_{jk}:\ \textit{Green-Lagrange strain tensor}} dx_j dx_k
\end{aligned} \tag{15}
$$

Assume that the deformation mapping $\varphi(\mathbf{x})$ has the form:

$$
\varphi(\mathbf{x}) = \mathbf{x} + \mathbf{u} \tag{16}
$$

where $\mathbf{u}$ is the *displacement field*. Then,

$$
\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j} \tag{17}
$$

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and the Green-Lagrange strain tensor becomes:

$$
\begin{aligned}
2\varepsilon_{ij} &= \left(\delta_{mi} + \frac{\partial u_m}{\partial x_i}\right)\left(\delta_{mj} + \frac{\partial u_m}{\partial x_j}\right) - \delta_{ij} \\
&= \cancel{\delta_{ij}} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j} - \cancel{\delta_{ij}}
\end{aligned}
\tag{18}
$$

$$
\text{Green-Lagrange strain tensor :} \quad \boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)} \tag{19}
$$

When the absolute values of the derivatives of the displacement field are much smaller than 1, their products (nonlinear part of the strain) are even smaller and we'll neglect them. We will make this assumption throughout this course. Mathematically:

$$
\frac{\partial u_i}{\partial x_j} \ll 1 \Rightarrow \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j} \sim 0 \tag{20}
$$

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We will define the *linear part* of the Green-Lagrange strain tensor as the *small strain tensor*:

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \tag{21}$$

Remarks:

- The strain tensor is symmetric
- Six independent components of strain: three normal $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$, and three shear $\varepsilon_{12} = \varepsilon_{21}, \varepsilon_{23} = \varepsilon_{32}, \varepsilon_{31} = \varepsilon_{13}$

Special cases: 1D, 2D

Outline

Given: $\varepsilon_{ij}, \mathbf{e}_i$ and a new basis $\tilde{\mathbf{e}}_k$, determine the components of strain in the new basis $\tilde{\varepsilon}_{kl}$

$$\tilde{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i}\right) \tag{22}$$

We want to express the quantities with tilde on the right-hand side in terms of their non-tilde counterparts. Start by applying the chain rule of differentiation:

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{u}_i}{\partial x_k}\frac{\partial x_k}{\partial \tilde{x}_j} \tag{23}$$

Transform the displacement components:

$$\mathbf{u} = \tilde{u}_m \tilde{\mathbf{e}}_m = u_l \mathbf{e}_l \tag{24}$$

$$\tilde{u}_m(\tilde{\mathbf{e}}_m \cdot \tilde{\mathbf{e}}_i) = u_l(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{25}$$

$$\tilde{u}_m \delta_{mi} = u_l(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{26}$$

$$\tilde{u}_i = u_l(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{27}$$

take the derivative of $\tilde{u}_i$ with respect to x_k, as required by equation (23):

$$\frac{\partial \tilde{u}_i}{\partial x_k} = \frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{28}$$

and take the derivative of the reverse transformation of the components of the position vector $\mathbf{x}$:

$$\mathbf{x} = x_j \mathbf{e}_j = \tilde{x}_k \tilde{\mathbf{e}}_k \tag{29}$$

$$x_j(\mathbf{e}_j \cdot \mathbf{e}_i) = \tilde{x}_k(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \tag{30}$$

$$x_j \delta_{ji} = \tilde{x}_k(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \tag{31}$$

$$x_i = \tilde{x}_k(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \tag{32}$$

$$\frac{\partial x_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{x}_k}{\partial \tilde{x}_j}(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) = \delta_{kj}(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) = (\tilde{\mathbf{e}}_j \cdot \mathbf{e}_i) \tag{33}$$

Replacing equations (28) and (33) in (23):

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{u}_i}{\partial x_k}\frac{\partial x_k}{\partial \tilde{x}_j} = \frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) \tag{34}$$

Replacing in equation (22):

$$\tilde{\varepsilon}_{ij} = \frac{1}{2}\Big[\frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) + \frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_j)(\tilde{\mathbf{e}}_i \cdot \mathbf{e}_k)\Big] \tag{35}$$

Exchange indices l and k in second term:

$$\begin{aligned} \tilde{\varepsilon}_{ij} &= \frac{1}{2}\Big[\frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) + \frac{\partial u_k}{\partial x_l}(\mathbf{e}_k \cdot \tilde{\mathbf{e}}_j)(\tilde{\mathbf{e}}_i \cdot \mathbf{e}_l)\Big] \\ &= \frac{1}{2}\Big(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l}\Big)(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) \end{aligned} \tag{36}$$

Or, finally:

$$\boxed{\tilde{\varepsilon}_{ij} = \varepsilon_{lk}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k)} \tag{37}$$

In other words, we obtain the same transformation equations as what we found for the stress tensor components. This confirms that $\varepsilon = \varepsilon_{ij}\mathbf{e}_i\mathbf{e}_j$ is a second-order tensor. We can therefore use all the mathematical machinery of transformation of second-order tensor components we derived for stresses: principal strains and directions, maximum shear stress and corresponding directions, Mohr's circle for 2D strain states, etc.