16.001 Unified Engineering
Materials and Structures

Constitutive Material Response: Stress-strain relations

Reading assignments: A & J 3, CDL 5.1, 5.2, 5.4, 5.10

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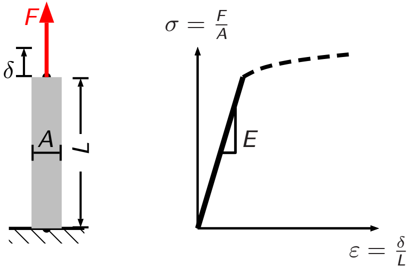
Outline

The mechanical response of a material is characterized experimentally. A mathematical relation between the components of the stress and the strain tensor is sought.

We have already discussed the uniaxial stress test and the general characteristics of the corresponding stress-strain curve relating the applied normal stress component σ_{11} and the corresponding normal strain component ε_{11}.

$$\sigma_{11} = E\varepsilon_{11}$$

We will look into this in more detail.



This expression describing the linear response of materials in the elastic regime is known as Hooke's law. This particular form applies only to a state of uniaxial stress. In general each and every component of the stress tensor σ_{ij} can depend on each and every component of the strain tensor ε_{ij}. We will generalize Hooke's law to general states of stress and strain.

Generalized Hooke's law I

A general linear relation between stress and strain components can be written as:

$$\boxed{\sigma_{ij} = C_{ijkl}\varepsilon_{kl}}$$

C_{ijkl} are the components of a fourth-order tensor known as the elasticity tensor.

Transformation of tensor components upon change of basis

Can show that basis transformation rules for fourth-order tensors are analogous to tensors of any order (including vectors), i.e., given the representation of tensor C in two different bases $\mathsf{e}_i, \tilde{\mathsf{e}}_j$:

$$\mathsf{C} = C_{ijkl}\mathsf{e}_i\mathsf{e}_j\mathsf{e}_k\mathsf{e}_l = \tilde{C}_{mnpq}\tilde{\mathsf{e}}_m\tilde{\mathsf{e}}_n\tilde{\mathsf{e}}_p\tilde{\mathsf{e}}_q$$

Taking dot products with basis vectors $\tilde{\mathsf{e}}_i$ four times, we find:

$$\boxed{\tilde{C}_{mnpq} = C_{ijkl}(\tilde{\mathsf{e}}_m \cdot \mathsf{e}_i)(\tilde{\mathsf{e}}_n \cdot \mathsf{e}_j)(\tilde{\mathsf{e}}_p \cdot \mathsf{e}_k)(\tilde{\mathsf{e}}_q \cdot \mathsf{e}_l)}$$

Symmetries of the elasticity tensor How many different coefficients are there in a fourth order-tensor? . However, for the special case of the elasticity tensor, there are symmetries that reduce their number significantly.

- Implications of symmetry of stress tensor.

$$\sigma_{ij} = \sigma_{ji}$$

$$C_{ijkl}\varepsilon_{kl} = C_{jikl}\varepsilon_{kl}, \Rightarrow$$

$$\left(C_{ijkl} - C_{jikl}\right)\varepsilon_{kl} = 0$$

$$\boxed{C_{ijkl} = C_{jikl}}$$

This reduces the number of coefficients to 54, since:

$$C \quad \overbrace{ijkl} \qquad \underbrace{\cancel{3x3}x3x3 = \cancel{81}}_{6x3x3=54}$$

- Implications of symmetry of strain tensor.

$$\varepsilon_{ij} = \varepsilon_{ji}$$

$$C_{ijkl}\varepsilon_{kl} = C_{ijkl}\varepsilon_{lk} = C_{ijlk}\varepsilon_{kl}, \Rightarrow$$

$$\left(C_{ijkl} - C_{ijlk}\right)\varepsilon_{kl} = 0$$

$$\boxed{C_{ijkl} = C_{ijlk}}$$

This reduces the number of coefficients to 36, since:

$$C \quad \overbrace{ijkl} \qquad \underbrace{\cancel{3x3}x\cancel{3x3} = \cancel{81}}_{6x6=36}$$

Generalized Hooke's law III

- Finally, from thermodynamics, it can be shown that in an elastic material the stresses are a state function of the strains, i.e. they derive from a thermodynamic potential (strain energy):

$$\sigma_{ij} = \frac{\partial \psi}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} \tag{1}$$

$$\frac{\partial^2 \psi}{\partial \varepsilon_{mn}\partial \varepsilon_{ij}} = \frac{\partial}{\partial \varepsilon_{mn}}\left(C_{ijkl}\varepsilon_{kl}\right) \tag{2}$$

$$C_{ijkl}\delta_{km}\delta_{ln} = \frac{\partial^2 \psi}{\partial \varepsilon_{mn}\partial \varepsilon_{ij}} \tag{3}$$

$$C_{ijmn} = \frac{\partial^2 \psi}{\partial \varepsilon_{mn}\partial \varepsilon_{ij}} \tag{4}$$

Assuming equivalence of the mixed partials:

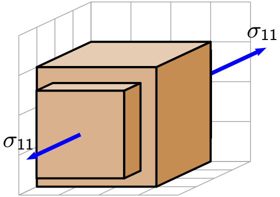
$$C_{ijkl} = \frac{\partial^2 \psi}{\partial \varepsilon_{kl}\partial \varepsilon_{ij}} = \frac{\partial^2 \psi}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}} = C_{klij} \tag{5}$$

This further reduces the number of material constants to 21. The most general anisotropic linear elastic material therefore has 21 material constants. We can write the stress-strain relations for a linear elastic material exploiting these symmetries as follows:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & symm & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \tag{6}$$

Let's revisit the uniaxial stress test and consider all the deformations it produces. For now, let's assume we are testing a material whose response is the same no matter the direction in which it is tested. We will call this material *isotropic*.

We observe that σ_{11} not only produces an elongation $\varepsilon_{11} = \frac{1}{E}(\sigma_{11})$ but also lateral contractions $\varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11} = -\nu\frac{1}{E}(\sigma_{11})$, where $\nu \equiv$ Poisson ratio is a material-dependent property. We also observe that no shear strains are produced for this isotropic material subject to a normal stress.



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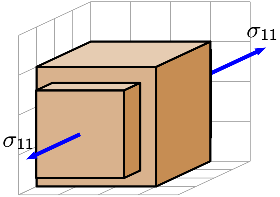
$$\varepsilon_{11} = \frac{\sigma_{11}}{E}$$



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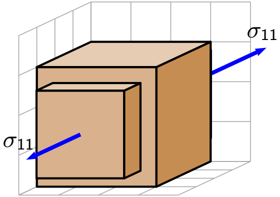
$$\varepsilon_{11} = \frac{\sigma_{11}}{E}$$
$$\varepsilon_{22} = \frac{1}{E}\left(-\nu\sigma_{11}\right)$$
$$\varepsilon_{33} = \frac{1}{E}\left(-\nu\sigma_{11}\right)$$



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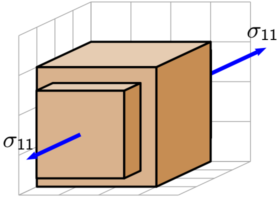
$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22}\right)\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left(-\nu\sigma_{11}\right)$$

$$\varepsilon_{33} = \frac{1}{E}\left(-\nu\sigma_{11}\right)$$

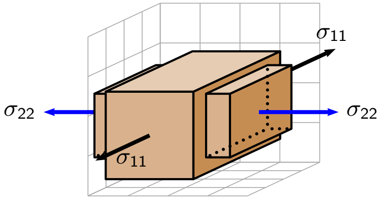


We now superimpose a uniaxial stress σ_{22}. Similarly, this only produces an elongation $\varepsilon_{22} = \frac{1}{E}(\sigma_{22})$ but also lateral contractions $\varepsilon_{11} = \varepsilon_{33} = -\nu\varepsilon_{22} = -\nu\frac{1}{E}(\sigma_{22})$,

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22}\right)\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11}\right)\right]$$

$$\varepsilon_{33} = \frac{1}{E}\left[-\nu\left(\sigma_{11} + \sigma_{22}\right)\right]$$

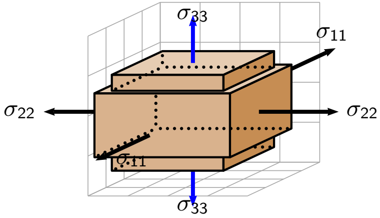


We now superimpose a uniaxial stress σ_{33}. Similarly, this only produces an elongation $\varepsilon_{33} = \frac{1}{E}(\sigma_{33})$ but also lateral contractions $\varepsilon_{11} = \varepsilon_{22} = -\nu\varepsilon_{33} = -\nu\frac{1}{E}(\sigma_{33})$,

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right]$$

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right]$$

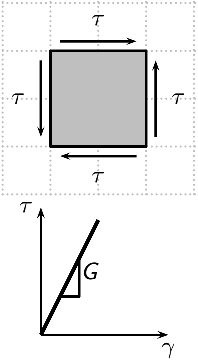


- We conceive a pure shear test as shown on the figure on the right.
- We apply a shear stress component $\sigma_{12} = \tau$ to a block of material and measure the total shear strain $2\varepsilon_{12} = \gamma$. No other strains are observed in an isotropic material.
- In the linear elastic range, the slope of the linear function relating these two quantities, is defined as the shear modulus G, i.e.:

$$\boxed{\sigma_{12} = \tau = G\gamma = G2\varepsilon_{12}}$$

- The same response is expected in any other of the shear directions, i.e.:

$$\boxed{\sigma_{23} = \tau = G\gamma = G2\varepsilon_{23}} \boxed{\sigma_{31} = \tau = G\gamma = G2\varepsilon_{31}}$$



General stress-strain relations for isotropic linear elastic materials: Hooke's Law I

With the results from these conceptual experimental tests and the elastic properties identified above, we can write the complete stress-strain relations in (6) in the following (inverted) form:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{7}$$

This suggests that isotropic linear elastic materials have three different constants: the elastic or Young's modulus E, the Poisson ratio ν and the shear modulus G. However, we can show that they are not all independent. The proof, is based on the realization, that the pure shear test is another test in disguise. All we need to do is to rotate the applied state of stress and observed strains to principal directions:

$$\begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix}, \begin{pmatrix} 0 & \frac{\gamma}{2} \\ \frac{\gamma}{2} & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{\gamma}{2} & 0 \\ 0 & -\frac{\gamma}{2} \end{pmatrix}$$

So in principal axes, our pure shear test has stress components

$$\sigma_{11} = \sigma_I = \tau, \sigma_{22} = \sigma_{II} = -\tau,$$

and strain components

$$\varepsilon_{11} = \varepsilon_I = \frac{\gamma}{2}, \varepsilon_{22} = \varepsilon_{II} = -\frac{\gamma}{2}$$

Applying Hooke's law to this set of normal stresses and strains:

$$\varepsilon_{11} = \frac{\gamma}{2} = \frac{1}{E}\left(\overbrace{\tau}^{\sigma_{11}} - \nu \overbrace{(-\tau)}^{\sigma_{22}}\right)$$

$$\gamma = \frac{2}{E}(1+\nu)\tau$$

General stress-strain relations for isotropic linear elastic materials: Hooke's Law III

But the coefficient relating γ and τ is $\frac{1}{G}$, from where we conclude that:

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

The second law of thermodynamics, which you will study in the spring, requires the shear modulus to be positive, or the material would require negative work (give me energy) when I try to deform it, instead of requiring positive work. This enforces a restriction on the possible values of the Poisson ratio $\boxed{\nu > -1}$. As we discussed in class, most materials have positive Poisson ratios.

General stress-strain relations for isotropic linear elastic materials: Hooke's Law IV

We can now rewrite:

Hooke's law in matrix form

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{8}$$

General stress-strain relations for isotropic linear elastic materials: Hooke's Law V

or just the plain equations exposing the actual couplings among normal and shear stresses and strains:

Hooke's law for isotropic linear elastic materials

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right] \tag{9}$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right] \tag{10}$$

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right] \tag{11}$$

$$2\varepsilon_{23} = \frac{1}{G}\sigma_{23} \tag{12}$$

$$2\varepsilon_{31} = \frac{1}{G}\sigma_{31} \tag{13}$$

$$2\varepsilon_{12} = \frac{1}{G}\sigma_{12} \tag{14}$$

We can now use this knowledge to explore special states of stress and strain

We will be using Hooke's law in a variety of special cases in which something is known about the state of stress and/or strain. It is interesting to see how the constitutive equations specialize to each one of those cases.

Uniaxial stress:

In this case, the only stress component is $\sigma_{11} = \sigma$ (for example), and all others are zero: $\sigma_{2j} = \sigma_{3j} = 0$. Using the constitutive laws, we obtain the only non-zero strains:

$$\varepsilon_{11} = \frac{1}{E}\sigma$$
$$\varepsilon_{22} = \frac{-\nu}{E}\sigma$$
$$\varepsilon_{33} = \frac{-\nu}{E}\sigma$$

Plane stress:

this means that stresses can only be in the plane, say e_1-e_2, i.e. $\sigma_{33} = \sigma_{31} = \sigma_{23} = 0$.

The constitutive equations simplify to:

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22})$$
$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11})$$
$$\varepsilon_{33} = \frac{-\nu}{E}(\sigma_{11} + \sigma_{22})$$
$$2\varepsilon_{12} = \frac{1}{G}\sigma_{12}$$

Inverting these, we get:

$$\sigma_{11} = \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22})$$
$$\sigma_{22} = \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11})$$
$$\varepsilon_{33} = \frac{-\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22})$$
$$\sigma_{12} = 2G\varepsilon_{12}$$

Plane strain:

this means that strains can only be in the plane, say e_1-e_2, i.e. $\varepsilon_{33} = \varepsilon_{31} = \varepsilon_{23} = 0$.

The constitutive equations simplify to:

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{33} + \sigma_{11}\right)\right]$$

$$0 = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right]$$

$$2\varepsilon_{12} = \frac{1}{G}\sigma_{12}$$

Inverting these, we get:

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{11} + \nu\varepsilon_{22}\right]$$

$$\sigma_{22} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{22} + \nu\varepsilon_{11}\right]$$

$$\sigma_{33} = \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{11} + \varepsilon_{22})$$

$$\sigma_{12} = 2G\varepsilon_{12}$$

Special states of stress and strain IV

Uniaxial strain:

this means that strains can only be in 1D, say e_1, i.e. $\varepsilon_{2j} = \varepsilon_{3j} = 0$.

The first three constitutive equations simplify to:

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right]$$
$$0 = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{33} + \sigma_{11}\right)\right]$$
$$0 = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right]$$

Inverting these, we get:

$$\sigma_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{11}$$
$$\sigma_{22} = \sigma_{33} = \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{11}$$

Bulk Modulus

Establishes a relation between the *hydrostatic stress* or pressure: $p = \frac{1}{3}\sigma_{kk}$ and the volumetric strain $\theta = \varepsilon_{kk}$.

$$p = K\theta \; ; \; K = \frac{E}{3(1-2\nu)} \tag{15}$$

To see this, add up the first three isotropic Hooke's constitutive equations in compliance form:

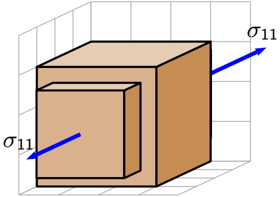
$$\underbrace{\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}_{\varepsilon_{kk}=\theta} = \frac{1}{E}\left[(\sigma_{11}+\sigma_{22}+\sigma_{33}) - \nu\left((\sigma_{22}+\sigma_{33}) + (\sigma_{11}+\sigma_{33}) + (\sigma_{11}+\sigma_{22})\right)\right]$$

$$= \frac{1}{E}\left[(\sigma_{11}+\sigma_{22}+\sigma_{33}) - 2\nu\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)\right]$$

$$= \frac{1}{E}\underbrace{(\sigma_{11}+\sigma_{22}+\sigma_{33})}_{\sigma_{kk}=3p}(1-2\nu)$$

$$\theta = \underbrace{\frac{3(1-2\nu)}{E}}_{1/K} p$$

Let's repeat our process of superimposing uniaxial stress states as we did for isotropic materials, but now, let's assume that the material is a composite with different fiber density in each direction. We thus expect a different stiffness in each direction as well. We will call this material *orthotropic*.
We observe that σ_{11} not only produces an elongation $\varepsilon_{11} = \frac{1}{E_1}(\sigma_{11})$ but also lateral contractions $\varepsilon_{22} = -\nu_{12}\varepsilon_{11}, \varepsilon_{33} = -\nu_{13}\varepsilon_{11}$. We also observe that no shear strains are produced as the load is aligned with the direction of the fibers (which are all aligned with the axes directions).



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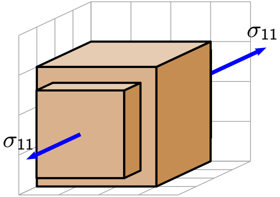
$$\varepsilon_{11} = \frac{\sigma_{11}}{E_1}$$



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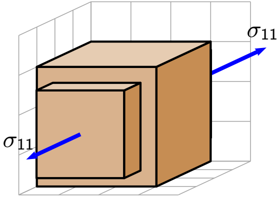
$$\varepsilon_{11} = \frac{\sigma_{11}}{E_1}$$
$$\varepsilon_{22} = -\nu_{12}\frac{\sigma_{11}}{E_1}$$



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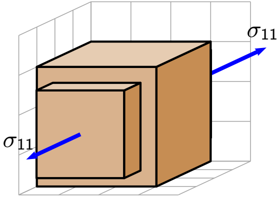
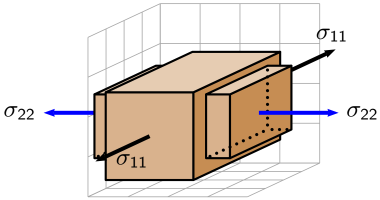
$$\varepsilon_{11} = \frac{\sigma_{11}}{E_1}$$
$$\varepsilon_{22} = -\nu_{12}\frac{\sigma_{11}}{E_1}$$
$$\varepsilon_{33} = -\nu_{13}\frac{\sigma_{11}}{E_1}$$



We now superimpose a uniaxial stress σ_{22}. Similarly, this produces an elongation $\varepsilon_{22} = \frac{\sigma_{22}}{E_2}$ (σ_{22}) but also lateral contractions
$\varepsilon_{11} = -\nu_{21}\varepsilon_{22}, \varepsilon_{33} = -\nu_{23}\varepsilon_{22}$

$$\varepsilon_{11} = \frac{\sigma_{11}}{E_1} - \nu_{21}\frac{\sigma_{22}}{E_2}$$
$$\varepsilon_{22} = -\nu_{12}\frac{\sigma_{11}}{E_1} + \frac{\sigma_{22}}{E_2}$$
$$\varepsilon_{33} = -\nu_{13}\frac{\sigma_{11}}{E_1} - \nu_{23}\frac{\sigma_{22}}{E_2}$$

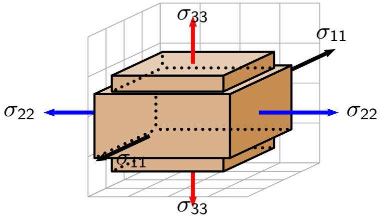


We now superimpose a uniaxial stress σ_{33}. Similarly, this produces an elongation $\varepsilon_{33} = \frac{\sigma_{33}}{E_3}(\sigma_{33})$ but also lateral contractions $\varepsilon_{11} = -\nu_{31}\varepsilon_{33}, \varepsilon_{22} = -\nu_{32}\varepsilon_{33}$

$$\varepsilon_{11} = \frac{\sigma_{11}}{E_1} - \nu_{21}\frac{\sigma_{22}}{E_2} - \nu_{31}\frac{\sigma_{33}}{E_3}$$

$$\varepsilon_{22} = -\nu_{12}\frac{\sigma_{11}}{E_1} + \frac{\sigma_{22}}{E_2} - \nu_{32}\frac{\sigma_{33}}{E_3}$$

$$\varepsilon_{33} = -\nu_{13}\frac{\sigma_{11}}{E_1} - \nu_{23}\frac{\sigma_{22}}{E_2} + \frac{\sigma_{33}}{E_3}$$



Reciprocity relations: As we discussed, the 1st law requires the relations above to be symmetric, e.g.

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}, \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}, \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}$$

Orthotropic linear elastic Hooke's law: Orthogonal composites

The shear response is also different in the three different orthogonal planes, as expected:

$$2\varepsilon_{23} = \frac{\sigma_{23}}{G_{23}},\ 2\varepsilon_{31} = \frac{\sigma_{31}}{G_{31}},\ 2\varepsilon_{12} = \frac{\sigma_{12}}{G_{12}}$$

Hooke's law for orthotropic materials in matrix form

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{16}$$

Remarks:

- When the loading and the material directions coincide, there is no coupling between shear and normal stresses and strains, and the shear response remains diagonal (i.e. no cross coupling among different shear stress and strain components).

Summary of Constitutive equations for orthotropic materials:

Hooke's law for orthotropic materials

$$\varepsilon_{11} = \frac{\sigma_{11}}{E_1} - \nu_{21}\frac{\sigma_{22}}{E_2} - \nu_{31}\frac{\sigma_{33}}{E_3} \tag{17}$$

$$\varepsilon_{22} = -\nu_{12}\frac{\sigma_{11}}{E_1} + \frac{\sigma_{22}}{E_2} - \nu_{32}\frac{\sigma_{33}}{E_3} \tag{18}$$

$$\varepsilon_{33} = -\nu_{13}\frac{\sigma_{11}}{E_1} - \nu_{23}\frac{\sigma_{22}}{E_2} + \frac{\sigma_{33}}{E_3} \tag{19}$$

$$2\varepsilon_{23} = \frac{\sigma_{23}}{G_{23}} \tag{20}$$

$$2\varepsilon_{31} = \frac{\sigma_{31}}{G_{31}} \tag{21}$$

$$2\varepsilon_{12} = \frac{\sigma_{12}}{G_{12}} \tag{22}$$

Summary of Constitutive equations for orthotropic materials:

Hooke's law for orthotropic materials

$$\varepsilon_{11} = \frac{1}{E_1}\left(\sigma_{11} - \nu_{12}\sigma_{22} - \nu_{13}\sigma_{33}\right) \tag{23}$$

$$\varepsilon_{22} = \frac{1}{E_2}\left(\sigma_{22} - \nu_{21}\sigma_{11} - \nu_{23}\sigma_{33}\right) \tag{24}$$

$$\varepsilon_{33} = \frac{1}{E_3}\left(\sigma_{33} - \nu_{31}\sigma_{11} - \nu_{32}\sigma_{22}\right) \tag{25}$$

$$2\varepsilon_{23} = \frac{\sigma_{23}}{G_{23}} \tag{26}$$

$$2\varepsilon_{31} = \frac{\sigma_{31}}{G_{31}} \tag{27}$$

$$2\varepsilon_{12} = \frac{\sigma_{12}}{G_{12}} \tag{28}$$