16.001 Unified Engineering
Materials and Structures

Structural theories for slender elements: Rod Theory

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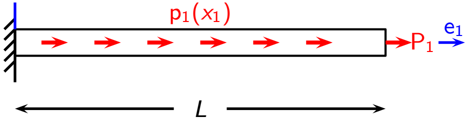
Outline



We will consider the structural response of slender structural elements subjected to axial loads only as shown in the figure, i.e. there are no transverse loadings. The only external loads possible in this case are either concentrated forces such as the load P_1, or distributed forces per unit length $p_1(x_1)$.

As we will do in any other structural theory, we will adopt a kinematic assumption, i.e. a restriction to the general form of the unknown displacement field $\mathrm{u}(\mathrm{x}) = u_i(x_j)\mathrm{e}_i$

Kinematic assumption

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1), \text{ cross sections remain planar} \tag{1}$$
$$u_2(x_1, x_2, x_3) = 0, \text{ no transverse deflections, cross sections deform rigidly} \tag{2}$$
$$u_3(x_1, x_2, x_3) = 0, \text{ no transverse deflections, cross sections deform rigidly} \tag{3}$$

We have simplified the description of the deformation to a single unknown function with a single independent variable $\bar{u}_1(x_1)$.

Strain field: The strain field then follows from the kinematic assumption from the strain-displacement relations we discussed last term $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$:

Strain field

$$\epsilon_{11}(x_1, x_2, x_3) = \bar{u}_1'(x_1)$$

and all the other strain components are zero. The assumption of allowing only rigid motions of the cross section implies that there cannot be any in-plane strains. This creates a state of uni-axial strain.

Constitutive law: We will assume a linear-elastic isotropic material and that the transverse stresses $\sigma_{22} = \sigma_{33} = 0$. By Hooke's law, the axial stress σ_{11} is given by:

$$\sigma_{11}(x_1, x_2, x_3) = E\epsilon_{11}(x_1, x_2, x_3)$$

Replacing the strain field for this case:

$$\sigma_{11}(x_1, x_2, x_3) = E\bar{u}_1'(x_1) \tag{4}$$

In other words, we are assuming a state of uni-axial stress.

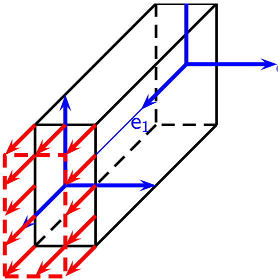
Remark: There is an intrinsic inconsistency in these assumptions, due to Poisson effect, we can either have $\sigma_{22} = \sigma_{33} = 0$ OR $\varepsilon_{22} = \epsilon_{33} = 0$, but not both. However, for most practical situations where Rod theory applies, this issue can be ignored.

Definition of stress resultant in the cross section: Since the stresses are uniform in the cross section, it is convenient to define their resultant effect as the total normal force they transmit at each cross section x_1 by simply integrating them in the area:

Stress resultant: Normal force

The *axial or normal force* is defined by the expression:

$$\boxed{N_1(x_1) = \int_A \sigma_{11}(x_1, x_2, x_3)dA} \qquad (5)$$



Constitutive law for the cross section: We can obtain a constitutive relation between the primary kinematic unknown $\bar{u}_1(x_1)$ and the primary kinetic (force) unknown $N_1(x_1)$ by combining equations (5) and (4):

$$N_1(x_1) = \int_{A(x_1)} E\,\bar{u}_1'(x_1)dA = \underbrace{\int_{A(x_1)} EdA}_{S(x_1)}\,\bar{u}_1'(x_1)$$

We will define:

$$S = \int_{A(x_1)} E(x_1, x_2, x_3)dA \tag{6}$$

as the *axial stiffness* of the beam, where we allow the Young's modulus to vary freely both in the cross section and along the axis of the beam, and we allow for non-uniform cross section geometries. In the case that the section is homogeneous in the cross section ($E = E(x_1, \cancel{x_2}, \cancel{x_3})$), we obtain: $S(x_1) = E(x_1)A(x_1)$. Further, if the section is uniform along x_1 and the material is homogeneous ($E = const$), we obtain: $S = EA$.

We can then write a constitutive relation between the axial force and the appropriate measure of strain for the beam:

Constitutive law for the cross section

$$N_1(x_1) = EA(x_1)\bar{u}_1'(x_1) \tag{7}$$

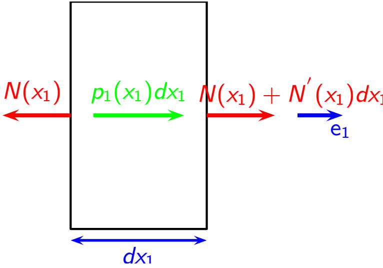
This very important expression defines a relation between the relevant kinetic variable for this problem $N_1(x_1)$ and the kinematic variable $\bar{u}_1(x_1)$. We see that it is a linear relation and that the proportionality constant $EA(x_1)$ plays the role of a stiffness. Further, we see that this stiffness has a contribution from the material (E) and another contribution from the geometry A. We will find a similar situation in other structural theories (e.g. beams bending, torsion).

Differential equation of equilibrium: In structural theories, we seek to impose equilibrium in terms of resultant forces (rather than at the material point as we did when we derived the equations of stress equilibrium). To this end, we consider the free body diagram of a slice of the beam as shown in the Figure. At x_1 the axial force is $N_1(x_1)$, at $x_1 + dx_1$, $N_1(x_1 + dx_1) = N_1(x_1) + N'(x_1)dx_1$. The distributed force per unit length $p_1(x_1)$ produces a force in the positive x_1 direction equal to $p_1(x_1)dx_1$. Equilibrium of forces in the e_1 direction requires:

$$-N_1(x_1) + p_1(x_1)dx_1 + N_1(x_1) + N_1'(x_1)dx_1 = 0$$

which implies:

$$\boxed{\frac{dN_1}{dx_1} + p_1 = 0} \qquad (8)$$



Putting it altogether: Governing equation: combining Equations (7) and (8).

Governing equation

$$\frac{d\left(EA(x_1)\bar{u}_1'(x_1)\right)}{dx_1} + p_1 = 0 \tag{9}$$

What principles does it enforce? compatibility, the constitutive law and equilibrium.

Boundary conditions:

- How many boundary conditions are required? It's a second order differential equation, thus it requires two boundary conditions
- What type of physical boundary conditions make sense for this problem and how are they expressed mathematically? The bar can be:
  - fixed, this implies that the displacement is specified to be zero

    $$\bar{u}_1 = 0$$

  - free (unloaded), which implies that:

    $$N_1 = EA\bar{u}_1' = 0, \Rightarrow \bar{u}_1' = 0$$

- subjected to a concentrated load P_1, which implies that:

$$N_1 = EA\bar{u}_1' = P_1$$

Revisit the Constitutive Law for uniaxial stress, allowing for thermal expansion effects:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} + \alpha \Delta\theta, \text{ inverting: } \sigma_{11} = E\left(\varepsilon_{11} - \alpha\Delta\theta\right)$$

Using assumptions from Rod Theory:

$$\sigma_{11}(x_1) = E\left(\varepsilon_{11}(x_1) - \alpha\Delta\theta(x_1)\right) = E\left(\bar{u}_1'(x_1) - \alpha\Delta\theta\right)$$

Replacing in the definition of the axial force:

Stiffness relation for rods subject to mechanical loads and thermal changes

$$N(x_1) = \int_{A(x_1)} \sigma_{11}(x_1) dA = \int_{A(x_1)} E\left(\bar{u}_1'(x_1) - \alpha\Delta\theta\right) dA$$

$$\boxed{N(x_1) = EA(x_1)\left(\bar{u}_1'(x_1) - \alpha\Delta\theta(x_1)\right)}$$

Combined with the equilibrium equation $N'(x_1) + p_1(x_1) = 0$, and the usual boundary conditions, we can solve for $\bar{u}_1'(x_1), N(x_1)$.