Moment of inertia of beam cross section

Reading:

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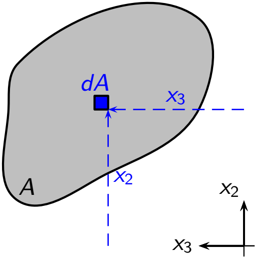
Outline

Geometric properties of 2D figures

- First moment of area: Center of area
- Second moment of area
- Parallel axis theorem



First moment of area Given an area A of any shape in the x_2-x_3 plane (as is the case for the cross section of a beam), the first moments of area with respect to the two axes are given respectively by:

$$\boxed{Q_2 = \int_A x_3 dA} \quad \boxed{Q_3 = \int_A x_2 dA}$$

They are a measure of the distribution of the area of the shape with respect to the given axes. The dimension of the moment of area is $[L^3]$, the SI units are m^3.

Moment of inertia of beam cross section II

Center of area or geometric center is defined as the location (x_2^G, x_3^G) such that:

$$Q_2 = \int_A x_3 dA = x_3^G A, \;\; Q_3 = \int_A x_2 dA = x_2^G A$$

i.e.

$$\boxed{x_3^G = \frac{Q_2}{A} = \frac{\int_A x_3 dA}{A}} \; \boxed{x_2^G = \frac{Q_3}{A} = \frac{\int_A x_2 dA}{A}}$$

Remarks

- Clearly if the axes are located on (x_2^G, x_3^G) the moments of area are zero, i.e.

$$Q_2^G = 0, \;\; Q_3^G = 0$$

- Any axis of symmetry of a 2D shape must pass through its Center of area

Moment of inertia of beam cross section III

For the remainder of the class, we will work with beam cross sections which have at least two orthogonal axes of symmetry.

Moment of Inertia or Second moment of area Given an area A of any shape in the x_2-x_3 plane (as is the case for the cross section of a beam), the second moments of area with respect to the two axes are given respectively by:

$$\boxed{I_{22} = \int_A x_3^2 dA} \quad \boxed{I_{33} = \int_A x_2^2 dA}$$

They are another (but different) measure of the distribution of the area of the shape with respect to the given axes. The dimension of the moment of inertia is $[L^4]$, its SI units are m^4.

Remarks

- As we saw in beam theory, the moment of inertia defines the geometric stiffness of a beam to bending loads. In Unified, we only consider bending in the plane x_1-x_2 where the beam stiffness is given by

$$\boxed{I = I_{33} = \int_A x_2^2 dA}$$

- Due to the symmetry of the stress distribution under pure bending loads (no net normal force), and the fact that we only consider homogeneous materials (bending of composites is discussed in 16.20), the distance x_2 in the expression above is the distance to the center of area, i.e. the neutral axis (the beam fibers with zero stress) is on the geometric center of the cross section.

Example

Moment of inertia of a circle of radius R with respect to centroidal axis

$$I = \int_A x_3^2 dA = \int_A (r \sin\theta)^2 dA = \int_0^{2\pi} \int_0^R (r \sin\theta)^2 (r dr d\theta) = \int_0^{2\pi} (\sin\theta)^2 \left(\int_0^R r^3 dr \right) d\theta = \frac{\pi}{4} R^4 \quad (1)$$

Theorem

Parallel axis theorem *Given the moment of inertia $I_{33}^G = I^G$ with respect to the the centroidal axis x_3 (i.e. the axis passing through its geometric center), the moment of inertia $I_{33} = I$ with respect to an axes x_3 separated from the original by a distance d_2 is:*

$$\boxed{I = I^G + Ad_2^2}$$

Proof.

$$I_{33} = \int_A x_2^2 dA = \int_A (x_2^G + d_2)^2 dA = \int_A \left(x_2^G\right)^2 dA + 2d_2 \underbrace{\int_A x_2^G dA}_{0} + \underbrace{\int_A d_2^2 dA}_{d_2^2 A}$$

$$= I + Ad_2^2 \quad (2)$$

q.e.d. □

This is very useful for computing the moments of inertia of beams with I, T, L shapes, etc.