16.001 Unified Engineering
Materials and Structures

Torsion

Reading assignments: CDL Ch. 6

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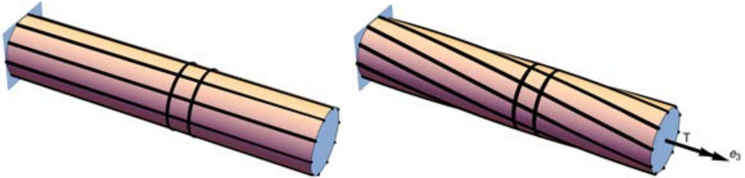
Outline:

General considerations I:



We will consider the structural response of slender structural elements subjected to torsional moments only as shown in the figure. The only external loads considered are either concentrated torques such as T, or distributed torques per unit length $t_3(x_3)$. We will also limit the discussion to circular bars with a solid or hollow cross section.

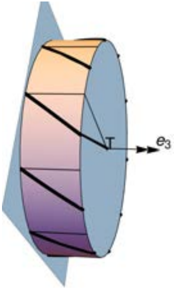
Kinematic assumptions I:

Following the observed deformation (Mathematica demonstration shown in class), we will develop a structural theory of torsion based on the following kinematic assumptions:

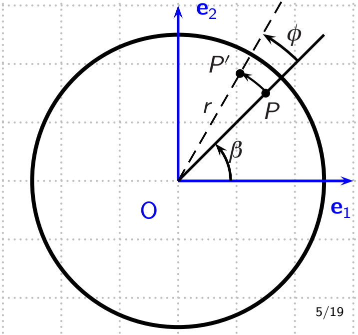
- Planar cross sections $x_3 = constant$ rotate as a rigid body about the axis of the cylindrical shaft.
- The angle of twist (rotation) of the cross sections will be a function of x_3 and denoted by $\phi(x_3)$.
- No other deformation will be allowed (e.g. axial stretching of the longitudinal fibers or warping of the cross section)

Kinematic assumptions II:

As can be seen in the figure and the demonstration, the only noticeable deformation resulting from these kinematic assumptions is a distortion of the square marks on the surface of the bar due to the relative rotation between the different sections. This is highlighted in the following figure



We now attempt to describe the deformation mathematically. To this end, we focus on the motion of an arbitrary point P at a cross section located at x_3 where the angle of twist is $\phi(x_3)$, whose in-plane undeformed coordinates are $x_1 = r\cos\beta, x_2 = r\sin\beta$. After the deformation (rigid rotation), the point occupies position P' of coordinates: $x_1' = r\cos(\beta+\phi), x_2' = r\sin(\beta+\phi)$ The displacement of point P is then (using trig. formulae followed by small ϕ assumption):



Kinematic assumptions III:

$$u_1 = x_1' - x_1 = r\left(\cos\left(\beta+\phi\right) - \cos\beta\right) = r\left(\cos\beta\overbrace{\cos\phi}^{\sim 1} - \sin\beta\overbrace{\sin\phi}^{\sim\phi} - \cos\beta\right)$$

$$u_2 = x_2' - x_2 = r\left(\sin\left(\beta+\phi\right) - \sin\beta\right) = r\left(\sin\beta\cos\phi + \cos\beta\sin\phi - \sin\beta\right)$$

$$\boxed{u_1 = -\underbrace{r\sin\beta}_{x_2}\phi(x_3) = -x_2\phi(x_3),\;\; u_2 = \underbrace{r\cos\beta}_{x_1}\phi(x_3) = x_1\phi(x_3)}$$

From the kinematic assumptions, we conclude that the most general form of the displacement field in this theory is:

$$\boxed{u_1(x_1,x_2,x_3) = -\phi(x_3)x_2} \tag{1}$$

$$\boxed{u_2(x_1,x_2,x_3) = \phi(x_3)x_1} \tag{2}$$

$$\boxed{u_3(x_1,x_2,x_3) = 0} \tag{3}$$

Kinematic assumptions IV:

We observe that from the perspective of kinematics, in this theory the determination of the full displacement field of the general theory of elasticity is reduced to the determination of a single scalar function of a single variable which describes the variation of the angle of twist along the axis of the shaft $\phi(x_3)$.

Strains in torsion theory:

As in other reduced or structural theories, the strains in torsion theory follow from the application of the definition of strain components to the assumed displacement field, (3):

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = \frac{\partial(-\phi(x_3)x_2)}{\partial x_1} = 0$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = \frac{\partial(\phi(x_3)x_1)}{\partial x_2} = 0$$

$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3} = \frac{\partial 0}{\partial x_3} = 0$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(-\phi(x_3) + \phi(x_3)) = 0$$

$$\boxed{\epsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = -\frac{1}{2}x_2\frac{d\phi}{dx_3}}$$

$$\boxed{\epsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}x_1\frac{d\phi}{dx_3}}$$

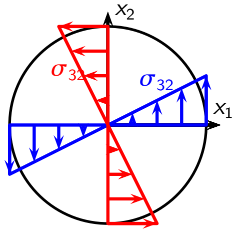
Stress-strain relations in torsion theory:

We will assume a linear, elastic, isotropic material (Hooke's law). Absence of normal strains: $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = 0$, implies absence of normal stresses: $\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$. This follows directly from Hooke's law. The shear stresses in an isotropic elastic model are directly related to the shear strains via the unique shear modulus G:

$$\sigma_{12} = 2G\epsilon_{12} = 0 \tag{4}$$

$$\boxed{\sigma_{13} = 2G\epsilon_{13} = -\frac{\cancel{2}}{\cancel{2}} G x_2 \frac{d\phi}{dx_3}} \tag{5}$$

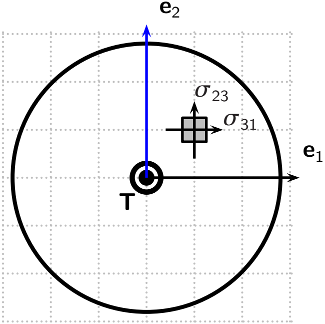
$$\boxed{\sigma_{23} = 2G\epsilon_{23} = \frac{\cancel{2}}{\cancel{2}} G x_1 \frac{d\phi}{dx_3}} \tag{6}$$



Note that the only stresses in this theory are shear stresses acting on the plane of direction x_3 (the cross section) and pointing in the direction tangential to the plane. Of course, these must be accompanied by shear stresses acting on the planes of normals x_1 and x_2 and pointing in direction x_3: $\sigma_{32} = \sigma_{23}, \sigma_{31} = \sigma_{13}$. Furthermore, Equations (5), (6) show that the distribution is linear in the radial direction.

Stress resultants in torsion theory: Internal torque I:

As in other structural theories discussed (rod, beam), we are interested in defining a resultant kinetic quantity in the cross section that aggregates the combined action of the internal stresses. The only existing stresses σ_{31}, σ_{23} can only contribute to resultant shear forces in directions x_1, x_2 (S_1, S_2), and a moment in direction x_3, M_3. These resultants are defined by the following expressions, see Figure:



$$\boxed{S_1(x_3) = \int_A \sigma_{31}(x_1, x_2, x_3) dA}$$

$$\boxed{S_2(x_3) = \int_A \sigma_{23}(x_1, x_2, x_3) dA}$$

$$\boxed{M_3(x_1) = \int_A (x_1 \sigma_{23}(x_1, x_2, x_3) - x_2 \sigma_{31}(x_1, x_2, x_3)) dA}$$

Stress resultants in torsion theory: Internal torque II:

Replacing the expressions for the stresses from Equations (5), (6):

$$S_1(x_3) = \int_A -Gx_2\phi'(x_3)dA = 0$$

$$S_2(x_3) = \int_A Gx_1\phi'(x_3)dA = 0$$

$$M_3(x_1) = \int_A (x_1 Gx_1\phi'(x_3) - x_2(-x_2 G\phi'(x_3)))dA = G\left(\int_A (x_1^2 + x_2^2)dA\right)\phi'(x_3)$$

$$= G\underbrace{\left(\int_A r^2 dA\right)}_{J=\frac{\pi R^4}{2}}\phi'(x_3)$$

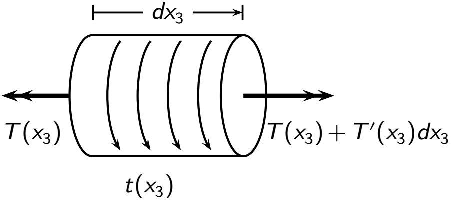
Stress resultants in torsion theory: Internal torque III:

And we obtain the internal torque-rate of twist relation for torsion:

$$\boxed{T(x_3) = M_3(x_3) = GJ\phi'(x_3)} \tag{7}$$

As with other structural theories, we obtain a stiffness relation between the metric of deformation appropriate to the theory, in this case the rate of twist $\phi'(x_3)$, and the metric of internal forcing, in this case, the internal torque T. The factor GJ in this linear relation is the torsional stiffness with contributions from the material (G), and geometry of the cross section J. As other structural stiffness relations, Equation (7) encodes the principles of compatibility and constitutive relations.
$J = \int_A r^2 dA$ represents the Polar moment of inertia which for circular cross sections can be readily found to be $J = \frac{\pi R^4}{2}$.

Equilibrium equation:



- Consider the free body diagram (FBD) of an infinitesimal segment of our shaft exposing the internal torques. The shaft is allowed to experience an externally-applied distributed torque per unit length $t(x_3)$. Equilibrium of moments in direction x_3 gives:

$$\sum M_3 = 0: \quad \cancel{-T} + \cancel{T} + T'(x_3)dx_3 + t(x_3)dx_3 = 0$$

$$\boxed{T'(x_3) + t(x_3) = 0} \tag{8}$$

Summary of equations I:

The governing equations for the torsion of circular shafts are:

$$\boxed{T'(x_3) + t(x_3) = 0} \qquad \text{(Equilibrium)} \tag{9}$$

$$\boxed{T(x_3) = GJ\phi'(x_3)} \qquad \text{(Compatibility, Constitutive)} \tag{10}$$

These can be combined into a single differential equation:

$$\boxed{(GJ\phi'(x_3))'(x_3) + t(x_3) = 0} \tag{11}$$

which is the (second-order) differential equation governing the angle of twist distribution in torsion theory.

Solution approach I:

As we can see, there are some common elements with other structural theories we have seen:

- The equilibrium equation is in this case a first-order ODE on a single unknown, the internal torque distribution $T(x_3)$. It can be solved independently if we know the value of T at some point in our shaft (typically one of the ends). In this case, the problem is statically determinate. The torque distibution obtained can then be inserted in the stiffness relation, which, in turn, is a first-order ODE that can be integrated to find the distribution of the angle of twist $\phi(x_3)$, provided we know its value at some point of the shaft (typically the other end).
- When the system is statically indeterminate, Equation (11) is used in combination with two kinematic boundary conditions to solve for $\phi(x_3)$. This can then be replaced in the stiffness relation to obtain $T(x_3)$.

Solution approach II:

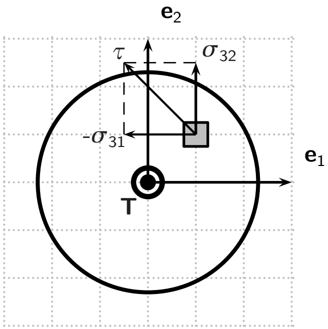
Computation of shear stresses: The stresses can be recovered from Equations (5) and (6) using the stiffness relation one more time as follows:

$$\sigma_{13} = -x_2 \overbrace{G\phi'(x_3)}^{\frac{T}{J}}, \quad \boxed{\sigma_{13} = -\frac{Tx_2}{J}} \tag{12}$$

$$\sigma_{23} = x_1 \overbrace{G\phi'(x_3)}^{\frac{T}{J}}, \quad \boxed{\sigma_{23} = -\frac{Tx_1}{J}} \tag{13}$$

The functional form of the stresses, as well as the axial symmetry, suggests a more physical expression in cylindrical instead of cartesian coordinates, which results in a shear stress component in the hoop direction $\sigma_{3\theta} = \tau$ that can be obtained as an in-plane resultant of the cartesian components as follows:

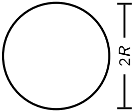
Solution approach III:

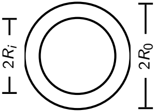


$$
\begin{aligned}
\tau &= \sqrt{\sigma_{13}^2 + \sigma_{23}^2} \\
&= \frac{T}{J}\sqrt{x_1^2 + x_2^2}
\end{aligned}
$$

$$
\boxed{\tau = \frac{Tr}{J}} \tag{14}
$$

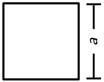
Limitations I:

- The torsion theory we developed assumes a circular cross section
- This includes the possibility of a hollow geometry (tube), which is very useful in practice for structural efficiency reasons. In this case, the polar moment of inertia J can be obtained from the additive decomposition of the integral.
- **Solid**: 

$J = \frac{\pi R^4}{2}$
- **Tube**: 

$J = \frac{\pi R_0^4}{2} - \frac{\pi R_i^4}{2}$
- Other cross sections require a more sophisticated analysis (discussed in 16.20) and usually lead to warping of the cross section in the axial direction.

Limitations II:

- For example for a **Square** cross section, it can be shown that: 

- $J = 0.141a^4$
- It is also found that open cross sections have very poor torsional stiffness