16.001 - Materials & Structures
Problem Set #6

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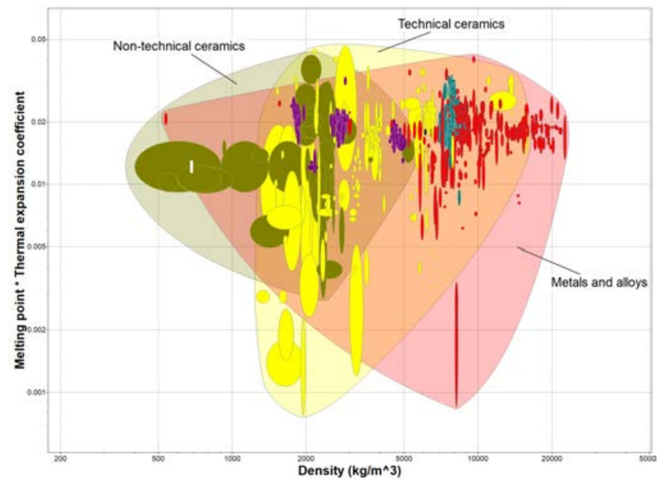
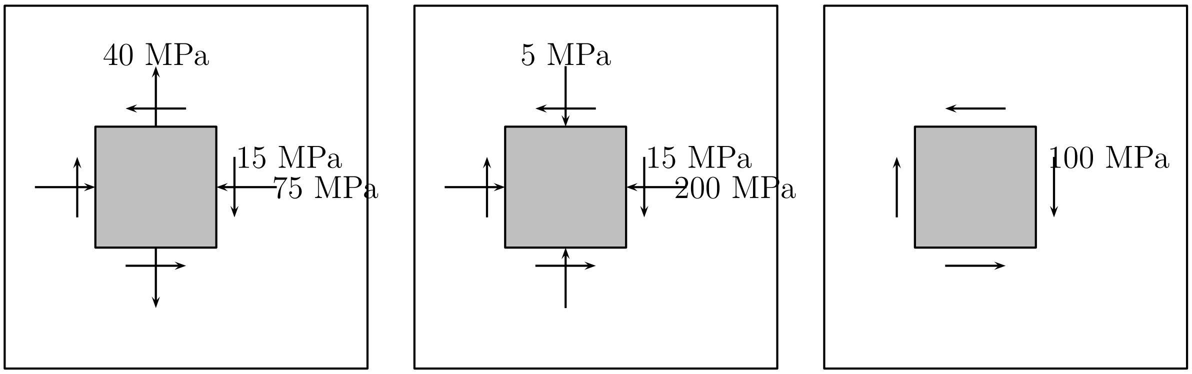
○ **Problems M-6.1** [5 points]

Metals and ceramics have a wide range of coefficient of thermal expansion (α) values and melting points (T_m); however, they all exhibit a similar amount of thermal expansion before they melt. Demonstrate this effect by generating a property diagram of density ρ vs. αT_m for bulk metals and ceramics and paste the property diagram below. You will need to use the "Advanced" y-axis function in the chart operation to compute a product of multiple material properties. What linear displacement would you expect to observe if you were to heat a 1 m long bar of some random metal or ceramic to its melting point?

Solution: According to this property diagram, most metals and ceramics have a thermal strain of 1 to 2% at their melting point. We therefore expect an unconstrained 1 m long bar of a random metal or ceramic to grow 1 mm due to thermal expansion before it starts to melt.



Problems M-6.2 [6 points]
The state of stress on the surface of an airplane fuselage at three different locations are represented on the elements shown below in Figure **??**. For each location: a) Find the principal stresses and the orientation of the element at which they are achieved. b) Find the maximum shear stress and orientation. At this orientation also find the normal stresses.



(a) Case 1 (b) Case 2 (c) Case 3

2.1 (2 points) Case 1

Solution:
To start this problem some useful equations to find the principal stresses and orientation are

$$\tilde{\sigma}_I, \tilde{\sigma}_{II} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} \tag{1}$$

$$\tan\ 2\alpha_p = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} \tag{2}$$

The first equation will give the principal stresses and the second equation when you solve for θ_p give the orientation. Where the $'$ denotes the value in the new orientation. Equations to find the maximum shear stress and orientation use the equations below.

$$\sigma_s = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} \tag{3}$$

$$\tilde{\sigma}_{11}(\alpha_s), \tilde{\sigma}_{22}(\alpha_s) = \frac{\sigma_{11} + \sigma_{22}}{2} \tag{4}$$

$$\tan\ 2\alpha_s = -\frac{\sigma_{11} - \sigma_{22}}{2\sigma_{12}} \tag{5}$$

Where Equation 3 gives the maximum shear stress, Equation 4 results in the normal stress at the orientation of maximum shear stress, and Equation 5 gives the orientation of maximum shear.

The original orientation has the stress state of

$$\sigma_{11} = -75\ MPa$$
$$\sigma_{22} = 40\ MPa$$
$$\sigma_{12} = -15\ MPa$$

a)
The principal stresses and proper orientation of the first location is solved with equations 1-2. The principal stress state has zero shear stress.

$$\sigma_I = 41.9\ MPa$$
$$\sigma_{II} = -76.9\ MPa$$
$$\tilde{\sigma}_{12}(\alpha_p) = 0$$
$$\alpha_p = 7.3^o$$

b)
The Maximum shear stress, the normal stresses, and the orientation are solved using equations 3-5 above.

$$\tilde{\sigma}_{11}(\alpha_s) = \tilde{\sigma}_{22}(\alpha_s) = -17.5\ MPa$$
$$\sigma_s = -59.4\ MPa$$
$$\alpha_s = -37.7^o$$

2.2 (2 points) Case 2

Solution:
The original orientation has the stress state of

$$\sigma_{11} = -200\ MPa$$
$$\sigma_{22} = -5\ MPa$$
$$\sigma_{12} = -15\ MPa$$

a)
The principal stresses and proper orientation of the second location is solve with equations 1-2. The principal stress state has zero shear stress.

$$\sigma_I = -3.9\ MPa$$
$$\sigma_{II} = -201.1\ MPa$$
$$\tilde{\sigma}_{12}(\alpha_p) = 0$$
$$\alpha_p = 4.4^o$$

b)
The Maximum shear stress, the normal stresses, and the orientation are solved using equations 3-5 above.

$$\tilde{\sigma}_{11}(\alpha_s) = \tilde{\sigma}_{22}(\alpha_s) = -102.5\ MPa$$
$$\sigma_s = -98.6\ MPa$$
$$\alpha_s = -40.6^o$$

2.3 (2 points) Case 3

Solution:
The original orientation has the stress state of

$$\sigma_{11} = 0\ MPa$$
$$\sigma_{22} = 0\ MPa$$
$$\sigma_{12} = -100\ MPa$$

a)
The principal stresses and proper orientation of the third location is solve with equations 1-2. The principal stress state has zero shear stress.

$$\sigma_I = 100\ MPa$$
$$\sigma_{II} = -100\ MPa$$
$$\tilde{\sigma}_{12}(\alpha_p) = 0$$
$$\alpha_p = 45^o$$

b)
The element is already at a state of maximum shear stress.

Problems M-6.3 [6 points]
A point on a thin plate is subjected to the two successive states of stress shown in Figure 2. Determine the resultant state of stress represented on the element oriented as shown on the right.

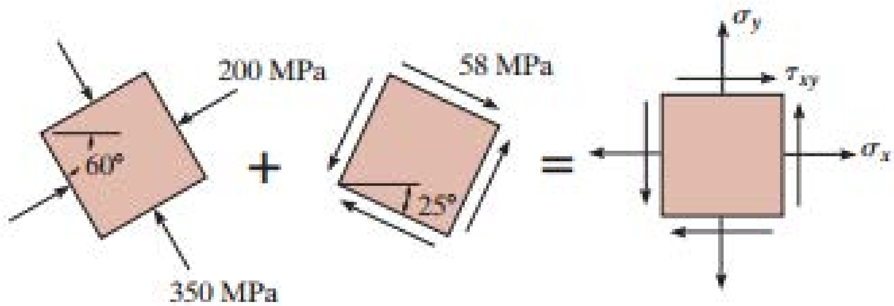


Figure 2: Addition of stress states

Solution:

To begin, transform the first to stress states to the final orientation. This could be done using the following equations.

$$\tilde{\sigma_{11}} = \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}cos(2\alpha) + \sigma_{12}sin(2\alpha)$$
$$\tilde{\sigma_{22}} = \frac{\sigma_{11}+\sigma_{22}}{2} - \frac{\sigma_{11}-\sigma_{22}}{2}cos(2\alpha) + \sigma_{12}sin(2\alpha)$$
$$\tilde{\sigma_{12}} = -\frac{\sigma_{11}-\sigma_{22}}{2}sin(2\alpha) + \sigma_{12}cos(2\alpha)$$

<u>State 1</u>

The given state is

$$\sigma_{11} = -350\ MPa$$
$$\sigma_{22} = -200\ MPa$$
$$\sigma_{12} = 0\ MPa$$
$$\alpha_1 = 60^\circ$$

yields the transformed state

$$\tilde{\sigma_{11}} = -237.5\ MPa$$
$$\tilde{\sigma_{22}} = -312.5\ MPa$$
$$\tilde{\sigma_{12}} = 65.0\ MPa$$

State 2

The given state is

$$\sigma_{11} = 0\ MPa$$
$$\sigma_{22} = 0\ MPa$$
$$\sigma_{12} = 58\ MPa$$
$$\alpha_1 = 25^\circ$$

yields the transformed state

$$\tilde{\sigma_{11}} = 44.4\ MPa$$
$$\tilde{\sigma_{22}} = -44.4\ MPa$$
$$\tilde{\sigma_{12}} = 37.28\ MPa$$

The final step is to add the two transformed states

$$\tilde{\sigma_{11}} = -193.1\ MPa$$
$$\tilde{\sigma_{22}} = -356.9\ MPa$$
$$\tilde{\sigma_{12}} = 102.0\ MPa$$

Problems M-6.4 [5 points]
The cylindrical pressure vessel has an inner radius of 1.25 m and a wall thickness of 20 mm. It is made from steel plates that are welded along the 45° seam. Determine the normal and shear stress component along this seam if the vessel is subjected to an internal pressure of 45 MPa.

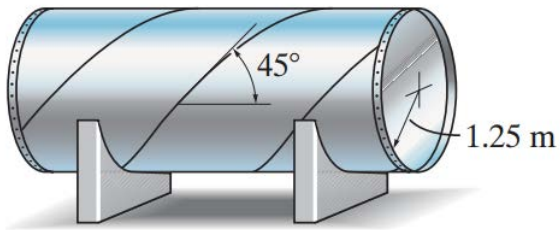


Figure 3: Pressure Vessel

Solution:

First, let's start by finding the stress state in the vessel at 0 incline (See Figure 4). The first normal stress we will find is σ_{hoop}. To do this cut the vessel as shown in Figure 5, and draw a FBD. Form this write the equation of equilibrium.

$$\sum F_2 = 2PrL - 2\sigma_{hoop}Lt = 0$$

where L is the length of the vessel and t is the thickness. From this we obtain

$$\sigma_{hoop} = \frac{pr}{t} = \frac{45(1.25)}{0.020} = 2812.5\ MPa$$

Next, find σ_{long}. Take a cut and draw a FBD (See Figure 6) and the write the equation of equilibirum.

$$\sum F_1 = P\pi r^2 - \sigma_{long}2\pi rt = 0$$

From this we obtain

$$\sigma_{long} = \frac{pr}{2t} = \frac{45(1.25)}{2(0.020)} = 1406.25\ MPa$$

Now the stress state along the weld needs to be calculated. This can be done using the equations for plane stress transformations

$$\sigma'_{11} = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2}cos(2\theta)$$
$$\sigma'_{22} = \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2}cos(2\theta)$$
$$\sigma'_{12} = -\frac{\sigma_{11} - \sigma_{22}}{2}sin(2\theta)$$

where $\sigma_{11} = \sigma_{long}$, $\sigma_{22} = \sigma_{hoop}$, and $\theta = 45^p$, we obtain the normal and shear stress along the weld

$$\sigma'_{11} = 2109.38\ MPa$$
$$\sigma'_{22} = 2109.38\ MPa$$
$$\sigma'_{12} = 703.125\ MPa$$

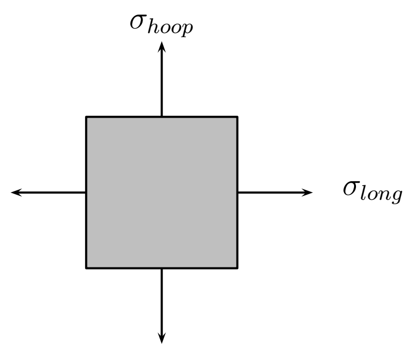


Figure 4: Stress State in Pressure Vessel

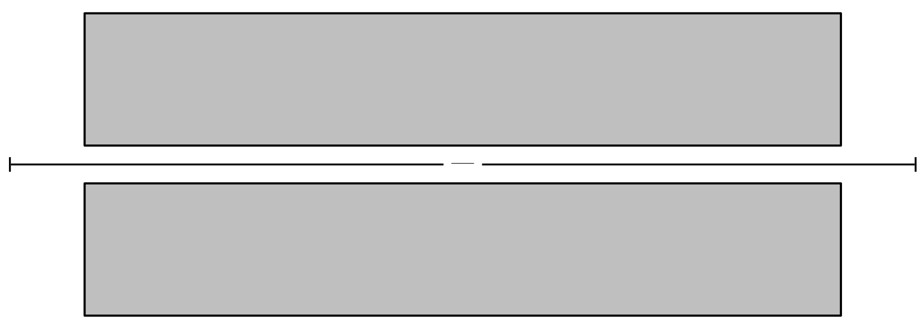


Figure 5: Cut to Find Hoop Stress

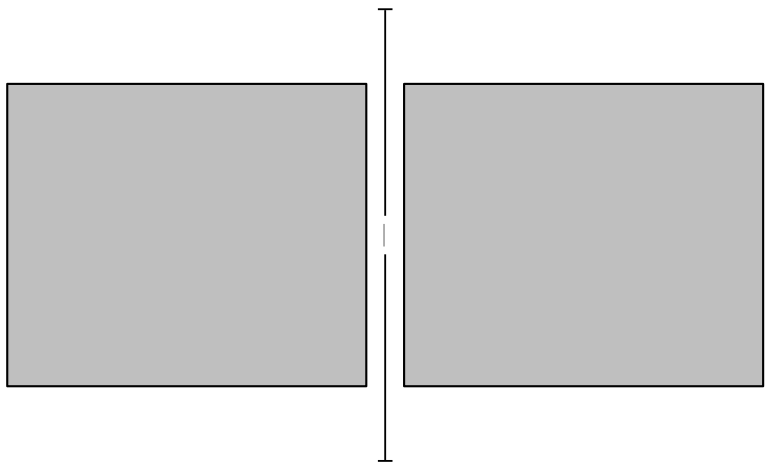


Figure 6: Cut to Find Longitudinal Stress

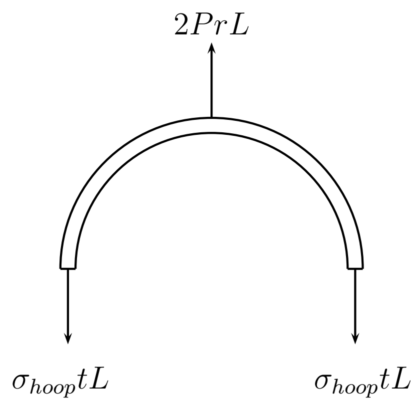


Figure 7: FBD for hoop stress

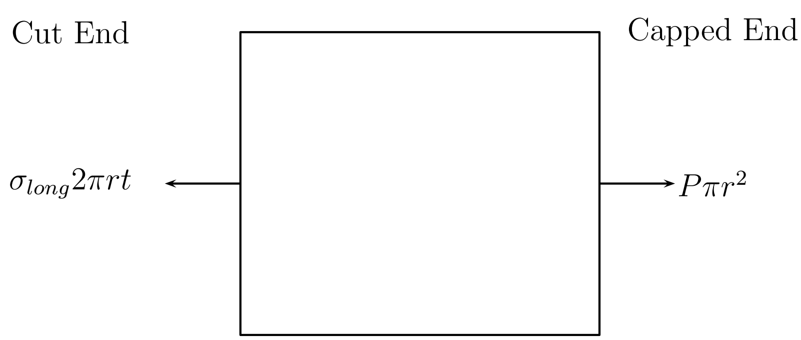


Figure 8: FBD for longitudinal stress

○ **Problems M-6.5** [6 points]
Three normal stress components $\sigma_{11} = 10, \sigma'_{11} = 5, \sigma''_{11} = -5$ (all in MPa) are given in three different directions $\mathbf{e}_1, \mathbf{e}'_1, \mathbf{e}''_1$ which are all at $60°$ apart from each other, i.e. $\widehat{\mathbf{e}_1\mathbf{e}'_1} = 60°$, $\widehat{\mathbf{e}'_1\mathbf{e}''_1} = 60°$.

5.1 (3 points) Determine all the stress components in the $\mathbf{e}_1, \mathbf{e}_2$ planes.

Solution: Apply repeatedly the Transformation equation for $\tilde{\sigma}_{11}(\alpha)$ to all values given. The LHS are known, the angles are known, the stress components σ_{12}, σ_{22} appear on the right hand side of the transformation equation and they are the unknowns:

$$\sigma'_{11} = \tilde{\sigma}_{11}(\pi/3) = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos(2\pi/3) + \sigma_{12} \sin(2\pi/3) \tag{6}$$

$$\sigma''_{11} = \tilde{\sigma}_{11}(2\pi/3) = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos(4\pi/3) + \sigma_{12} \sin(4\pi/3) \tag{7}$$

$$\tag{8}$$

Replacing the known values on the left and the right:

$$5 = \frac{10 + \sigma_{22}}{2} + \frac{10 - \sigma_{22}}{2} \left(\frac{-1}{2}\right) + \sigma_{12} \left(\frac{\sqrt{3}}{2}\right) \tag{9}$$

$$-5 = \frac{10 + \sigma_{22}}{2} + \frac{10 - \sigma_{22}}{2} \left(\frac{-1}{2}\right) + \sigma_{12} \left(-\frac{\sqrt{3}}{2}\right) \tag{10}$$

$$\tag{11}$$

We obtain a 2x2 system on σ_{22}, σ_{12}. Solving, we get:

$$\sigma_{22} = -\frac{10}{3}\text{MPa}, \sigma_{12} = \frac{10\sqrt{3}}{3}\text{MPa}$$

5.2 (1 point) Determine the principal stresses and directions

Solution: The mean stress is:

$$\sigma_{avg} = \frac{\sigma_{11} + \sigma_{22}}{2} = \frac{10}{3}\text{MPa}$$

The radius:

$$R = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} \sim 8.82\text{MPa}$$

The principal stresses:

$$\sigma_{I,II} = \sigma_{ave} \pm R \sim (3.33 \pm 8.82)\text{MPa} = 12.15\text{MPa}, -5.49\text{MPa}$$

The principal directions:

$$\tan 2\alpha_p = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} = \frac{\sqrt{3}}{2}, \alpha_{pI} = 20.45^\circ, \alpha_{pII} = 110.4^\circ$$

5.3 (2 points) Draw Mohr's circle for the state of stress corresponding to the given three normal stresses.

Solution: