16.001 - Materials & Structures
Problem Set #12

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Question	Points
1	0
2	0
3	30
4	20
Total:	50

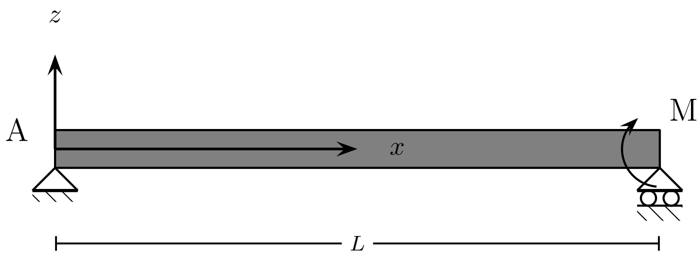

Figure 1: Simply supported beam

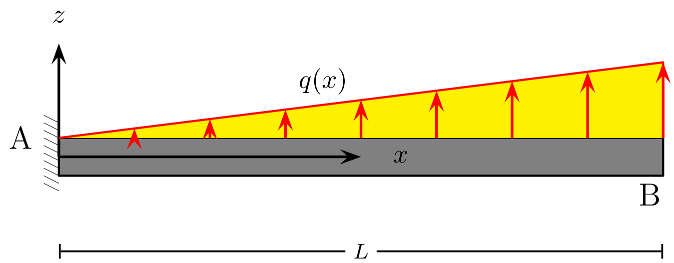

Figure 2: Cantilever beam

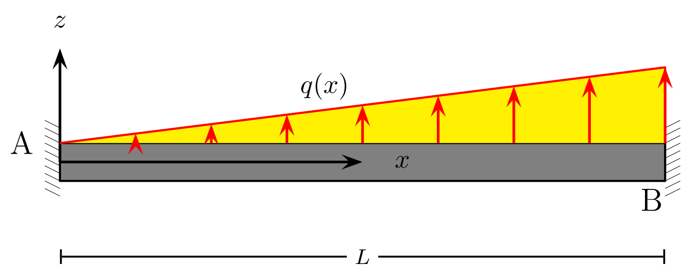

Figure 3: Clamped beam

Problems M-12.1 [0 points]

Consider the beams shown in figures 1-4. In all cases, the beams have a length L, a rectangular cross section of width b and height h, and are made of a material with Young's modulus E. For the beams with a linear variation in the distributed load, the equation describing the load is:

$$q(x) = q_o \frac{x}{L} \tag{1}$$

Please do the following for each of the beams:

1.1 (5 points) State and explain the boundary conditions.

1.2 (5 points) Obtain the moment(M), shear(S), deflection (w), and slope(w') distribution along the length of the beam. In each case, try to find the most efficient approach, explain which equations you use and why. (hint. Take into account of whether or not you can solve for the shear and moment without the use of the

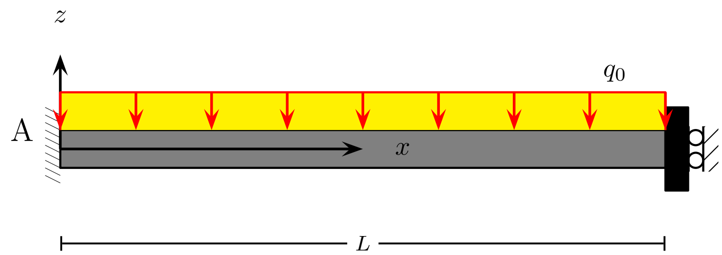

Figure 4

moment-curvature relation). Also, state whether the beam is statically determinate or statically indeterminate.

1.3 (5 points) Find the point-wise stress distribution $\sigma_{11}(x, z)$ along with its maximum value and the location it occurs.

Solution: Beam 1

Boundary Conditions

The boundary conditions for this beam are:

$$w(0) = 0$$
$$w(L) = 0$$
$$M(0) = 0$$
$$M(L) = -M$$

These boundary conditions mean no deflection at either end, that the left support the beam is free to rotate, and a moment to

Determining Distributed Fields

This problem is statically determinate, so the most efficient way to solve this problem is to start with the equilibrium equations instead of the fourth order ODE governing simple beam theory. Using the equations:

$$S'(x) = 0$$
$$M'(x) = -S(x)$$
$$M''(x) = 0$$

and integrating twice

$$M(x) = Ax + B$$

applying the two moment boundary conditions.

$$B.C.\ 3 \rightarrow B = 0$$
$$B.C.\ 4 \rightarrow A = \frac{-M}{L}$$

So the equations for Shear and Moment become

$$S(x) = \frac{M}{L}$$
$$M(x) = \frac{-M}{L}x$$

Next, using the moment-curvature equation find the slope and deflection fields.

$$EIw''(x) = M(x) = \frac{-M}{L}x$$

Integrating twice yields:

$$w(x) = \frac{1}{EI}\left[\frac{-M}{6L}x^3 + Cx + D\right]$$

Applying the BCs

$$B.C.\ 1 \rightarrow D = 0$$
$$B.C.\ 2 \rightarrow C = \frac{ML}{6}$$

Thus the slope and deflection equations become

$$w(x) = \frac{M}{6EI}\left[Lx - \frac{x^3}{L}\right]$$
$$w'(x) = \frac{M}{6EI}\left[L - \frac{3x^2}{L}\right]$$

Stress Distribution The stress distribution can be given by the equation:

$$\sigma_{11}(x, z) = \frac{-M(x)z}{I}$$

Plugging in the equation for the moment

$$\sigma_{11}(x, z) = \frac{M}{LI}xz$$

The equation is linear so the maximum value must be at the boundary in the x direction. Checking the bounds, the maximum value is:

$$\sigma_{11_{max}}(x = L, z = \pm\frac{h}{2}) = \pm\frac{Mh}{2I}$$

Solution: Beam 2

Boundary Conditions

The boundary conditions for this beam are:

$$\begin{aligned} w(0) &= 0 \\ w'(0) &= 0 \\ M(L) &= 0 \\ S(L) &= 0 \end{aligned}$$

These boundary conditions mean no deflection or rotation at the left end and the right end is free to rotate and deflect

Determining Distributed Fields

This problem is statically determinate, so the most efficient way to solve this problem is to start with the equilibrium equations instead of the fourth order ODE governing simple beam theory. Using the equations:

$$\begin{aligned} S'(x) &= q(x) \\ M'(x) &= S(x) \end{aligned}$$

integrating once

$$\begin{aligned} S'(x) &= -q_0 \frac{x}{L} \\ S(x) &= -q_0 \frac{x^2}{2L} + A \end{aligned}$$

applying the shear BC 4 to solve for A

$$\begin{aligned} A &= \frac{q_0 L}{2} \\ S(x) &= \frac{q_0}{2}\left[L - \frac{x^2}{L}\right] \end{aligned}$$

integrating again and applying BC 3

$$\begin{aligned} M'(x) &= \frac{q_0}{2}\left[\frac{x^2}{L} - L\right] \\ M(x) &= \frac{q_0}{2}\left[\frac{x^3}{3L} - Lx\right] + B \\ BC\ 3 \rightarrow\ B &= \frac{q_0 L^2}{3} \\ M(x) &= q_0\left[\frac{x^3}{6L} - \frac{L}{2}x + \frac{L^2}{3}\right] \end{aligned}$$

Next, using the moment-curvature equation find the slope and deflection fields.

$$EIw''(x) = M(x) = q_0\left[\frac{x^3}{6L} - \frac{L}{2}x + \frac{L^2}{3}\right]$$

Integrating twice yields:

$$w(x) = \frac{q_0}{EI}\left[\frac{x^5}{120L} - \frac{L}{12}x^3 + \frac{L^2}{6}x^2\right] + Cx + D$$

Applying the BCs

$$B.C.\ 1 \rightarrow D = 0$$
$$B.C.\ 2 \rightarrow C = 0$$

Thus the slope and deflection equations become

$$w(x) = \frac{q_0}{EI}\left[\frac{x^5}{120L} - \frac{L}{12}x^3 + \frac{L^2}{6}x^2\right]$$
$$w'(x) = \frac{q_0}{EI}\left[\frac{x^4}{24} - \frac{L}{4}x^2 + \frac{L^2}{3}x\right]$$

Stress Distribution

Next, plugging in our equation for the moment into the stress relation

$$\sigma_{11}(x,z) = \frac{-q_0 z}{I}\left[\frac{x^3}{6L} - \frac{L}{2}x + \frac{L^2}{3}\right]$$

The equation has local maximum/minimums at x = $\pm L$. Since there are no local max/mins inside the domain of the beam we just need to check the values at the boundaries. Doing this will yield:

$$\sigma_{11_{max}}(x=0, z=\pm\frac{h}{2}) = \frac{\mp q_0 h L^2}{6I}$$

Solution: Beam 3

Boundary Conditions

The boundary conditions for this beam are:

$$w(0) = 0$$
$$w'(0) = 0$$
$$w(L) = 0$$
$$w'(L) = 0$$

These boundary conditions mean no deflection or rotation at either end of the beam

Determining Distributed Fields

This problem is statically indeterminate, so the most efficient way to solve this problem is to start with the fourth order ODE governing simple beam theory:

$$EIw''''(x) = q(x) = q_0 \frac{x}{L}$$

Integrating 4 times to get the general form of the solution for w(x)

$$EIw(x) = q_0 \frac{x^5}{120L} + A\frac{x^3}{6} + B\frac{x^2}{2} + Cx + D$$

Applying the boundary conditions

$$\begin{aligned} BC\ 1 &\rightarrow\ D = 0 \\ BC\ 2 &\rightarrow\ C = 0 \\ BC\ 3\ \&\ 4 &\rightarrow\ A = \frac{-3Lq_0}{20},\ B = \frac{L^2 q_0}{30} \end{aligned}$$

Thus the equations for deflection, slope, moment, and shear become:

$$\begin{aligned} w(x) &= \frac{q_0}{EI}\left[\frac{x^5}{120L} - \frac{3L}{120}x^3 + \frac{L^2}{60}x^2\right] \\ w'(x) &= \frac{q_0}{EI}\left[\frac{x^4}{24L} - \frac{3L}{40}x^2 + \frac{L^2}{30}x\right] \\ M(x) = EIw''(x) &= q_0\left[\frac{x^3}{6L} - \frac{3L}{20}x + \frac{L^2}{30}\right] \\ S(x) = -M'(x) &= q_0\left[\frac{3L}{20} - \frac{x^2}{2L}\right] \end{aligned}$$

Stress Distribution

Next, plugging in our equation for the moment into the stress relation

$$\sigma_{11}(x,z) = \frac{-q_0 z}{I}\left[\frac{x^3}{6L} - \frac{3L}{20}x + \frac{L^2}{30}\right]$$

The equation has local maximum/minimums at x = $\pm\sqrt{\frac{6}{20}}L$. The location x = $\sqrt{\frac{6}{20}}L$ is within the domain, so the value there needs to be compared to the bounds. Doing this will yield:

$$\sigma_{11_{max}}(x = L, z = \pm\frac{h}{2}) = \frac{\mp q_0 h L^2}{40I}$$

Solution: Beam 4

Boundary Conditions

The boundary conditions for this beam are:

$$\begin{aligned} w(0) &= 0 \\ w'(0) &= 0 \\ w'(L) &= 0 \\ S(x) &= 0 \end{aligned}$$

These boundary conditions mean no deflection or rotation at the the left end, no rotation at the right end, and the beam being free to deflect at the right end.

Determining Distributed Fields

This problem is statically indeterminate, so the most efficient way to solve this problem is to start with the fourth order ODE governing simple beam theory:

$$EIw''''(x) = q(x) = -q_0$$

Integrating 4 times to get the general form of the solution for w(x)

$$EIw(x) = -\frac{q_0}{24}x^4 + A\frac{x^3}{6} + B\frac{x^2}{2} + Cx + D$$

Applying the boundary conditions

$$\begin{aligned} BC\ 1 &\rightarrow\ D = 0 \\ BC\ 2 &\rightarrow\ C = 0 \\ BC\ 4 &\rightarrow\ A = q_0 L \\ BC\ 3 &\rightarrow\ B = -\frac{L^2 q_0}{3} \end{aligned}$$

Thus the equations for deflection, slope, moment, and shear become:

$$\begin{aligned} w(x) &= \frac{q_0}{EI}\left[\frac{-x^4}{24} + \frac{L}{6}x^3 - \frac{L^2}{6}x^2\right] \\ w'(x) &= \frac{q_0}{EI}\left[\frac{-x^3}{6} + \frac{Lx^2}{2} - \frac{L^2}{3}x\right] \\ M(x) &= EIw''(x) = q_0\left[\frac{-x^2}{2} + Lx - \frac{L^2}{3}\right] \\ S(x) &= M'(x) = q_0(x - L) \end{aligned}$$

Stress Distribution

Next, plugging in our equation for the moment into the stress relation

$$\sigma_{11}(x,z) = \frac{-q_0 z}{I}\left[\frac{-x^2}{2} + Lx - \frac{L^2}{3}\right]$$

The equation has local maximum/minimums at x = L. Since there are no local max/mins inside the domain of the beam we just need to check the values at the boundaries. Doing this will yield:

$$\sigma_{11_{max}}(x=0, z=\pm\frac{h}{2}) = \frac{\pm q_0 h L^2}{6I}$$

Problems M-12.2 [0 points]
The built-in beam shown in Figure 5 has a length L, and bending stiffness EI.

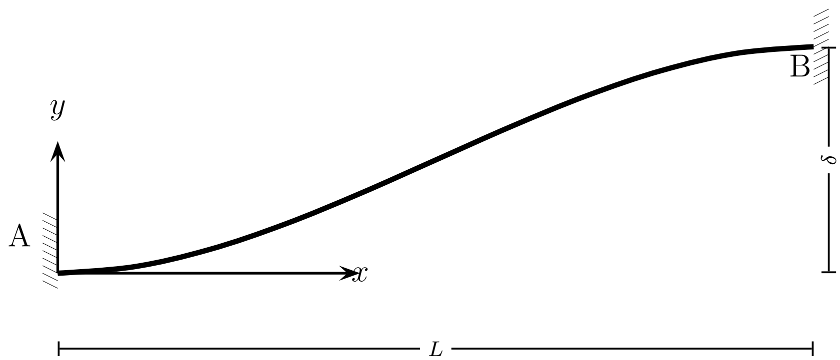

Figure 5: built-in beam subject to end deflection δ

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2.1 (10 points) Write down the equations governing the distribution of the following functions: deflection $u(x)$, bending moment $M(x)$ and shear $S(x)$. Indicate what principle each equation represents. Show that you can combine these equations to obtain a single ordinary differential equation governing beam bending which reads as follows:

$$EIu^{(IV)}(x) = 0$$

Solution:

$$\text{equilibrium of moments: } M' + S = 0$$
$$\text{equilibrium of transverse forces: } S' + q = 0$$
$$\text{compatibility and constitutive law: } M = EIu''$$

Combine the three, use $q = 0$ to obtain sought result.

2.2 (20 points) Write down the boundary conditions for this problem and use them to find the solution for the deflection of the beam $u(x)$, the moment $M(x)$ and the shear $S(x)$. You should obtain the following result:

$$u(x) = \delta \left(\frac{x}{L}\right)^2 \left[3 - \frac{2x}{L}\right]$$
$$u'(x) = \frac{6\delta}{L}\left[\frac{x}{L}\right](1 - \frac{x}{L})$$
$$M(x) = EI\frac{6\delta}{L^2}(1 - \frac{2x}{L})$$
$$S(x) = -M'(x) = 12\delta\frac{EI}{L^3}$$

Solution: The boundary conditions for this problem are

$$u(0) = u'(0) = u'(L) = 0$$
$$u(L) = \delta$$

Integration of the governing equation

$$EIu'''(x) = C$$
$$EIu''(x) = Cx + D$$
$$EIu'(x) = C\frac{x^2}{2} + Dx + F$$
$$EIu(x) = C\frac{x^3}{6} + D\frac{x^2}{2} + Fx + G$$

Applying the boundary conditions

$$u'(0) = 0 \rightarrow F = 0$$
$$u(0) = 0 \rightarrow G = 0$$
$$u'(L) = 0 \rightarrow D = -\frac{CL}{2} = \frac{6\delta EI}{L^2}$$
$$u(L) = \delta \rightarrow C = \frac{-12\delta EI}{L^3}$$

Substituting the integration coefficients back into the equation for the deflection field

$$u(x) = \delta \left(\frac{x}{L}\right)^2 \left[3 - \frac{2x}{L}\right]$$
$$u'(x) = \frac{6\delta}{L}\left[\frac{x}{L}\right](1 - \frac{x}{L})$$
$$M(x) = EI\frac{6\delta}{L^2}(1 - \frac{2x}{L})$$
$$S(x) = -M'(x) = 12\delta\frac{EI}{L^3}$$

2.3 (5 points) Interpret the result. Specifically, explain the shape of the shear and moment distributions.

Solution: A vertical reaction will appear at $x = L$ to support the imposed displacement δ. There is no q to modify this shear, thus the shear is constant. The reaction at $x = 0$ will point down. Both reactions will cause a moment. The change of curvature indicates the signs of the moments at the extreme ends.

There will be no curvature and thus no moment at $x = L/2$, etc.

2.4 (5 points) In the rest of this problem, we will explore how to obtain the same solution by exploiting the principle of superposition. Below, you are given the solution fields for various statically-determinate beams.

Choose and adequate subset of those solutions that you could combine to obtain the solution to the problem above. State and justify your choices.

Solution: One possible combination is solutions 1 and 3 from the appendix. Solution 1 satisfies all the boundary conditions of our problem except that it violates the zero-rotation BC at $x = L$. Solution 3 satisfies the BCs at $x = 0$, gives the possibility to create an arbitrary rotation that could cancel the undesirable rotation at $x = L$ in solution 1, by proper selection of the moment M_0.

2.5 (5 points) Explain the procedure by which you will use superposition to obtain the solution to the indeterminate problem using the two determinate problems

Solution:

- Find the value of M_0 that would eliminate the spurious rotation of solution 1 at $x = L$
- Use this value in solution 2 and add up the two solutions to obtain the desired result

2.6 (20 points) Execute the procedure and show that you obtain the same result as in Part (2)

Solution:

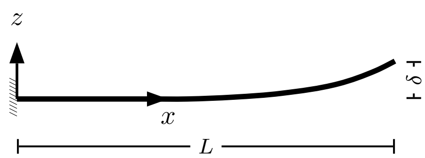
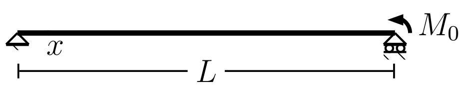
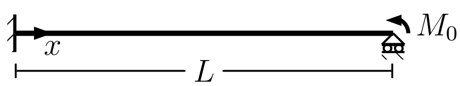
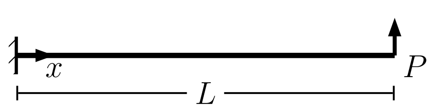
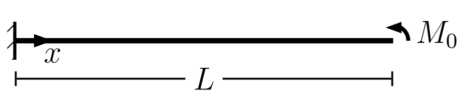
$$
\begin{aligned}
u'(L) = 0 &= u'^{①}(L, \delta) + u'^{③}(L, M_0) \\
&= 0 = \frac{\delta}{L^3} \frac{3}{2} L^2 + \frac{1}{4} \frac{M_0 L}{EI} \\
\rightarrow M_0 &= -6\delta \frac{EI}{L^2}
\end{aligned}
$$

Substituting this value for M_0 into the deflection field for solution 3 and add it to the deflection field of solution 1 to get the final deflection field.

$$
\begin{aligned}
u(x) &= u^{①}(x;\delta) + u^{③}(x; M_0 = -6\delta\frac{EI}{L^2}) \\
&= \frac{1}{2}\delta(\frac{x}{L})^2(3-\frac{x}{L}) + \left(-\frac{6\delta EI}{L^2}\right)\frac{1}{4EI}x^2(\frac{x}{L}-1) \\
&= \delta\left(\frac{x}{L}\right)^2\left[3-2\frac{x}{L}\right]
\end{aligned}
$$

Once again, from the deflection field you can obtain the rotation, moment, and shear fields, which are:

$$
\begin{aligned}
u'(x) &= \frac{6\delta}{L}\left[\frac{x}{L}\right](1-\frac{x}{L}) \\
M(x) &= EI\frac{6\delta}{L^2}(1-\frac{2x}{L}) \\
S(x) &= -M'(x) = 12\delta\frac{EI}{L^3}
\end{aligned}
$$

1)		$\boxed{u(x) = -\frac{3\delta}{L^3}\left[\frac{x^3}{6} - \frac{Lx^2}{2}\right]}$ $\boxed{u'(x) = \frac{\delta}{L^3}\left[-\frac{3}{2}x^2 + 3Lx\right]}$ $\boxed{M(x) = \frac{3EI\delta}{L^3}\left[L - x\right]}$ $\boxed{S(x) = \frac{3EI\delta}{L^3}}$
2)		$\boxed{u(x) = \frac{M_0 L^2}{6\mathrm{EI}}(\frac{x}{L})\left[(\frac{x}{L})^2 - 1\right]}$ $\boxed{u'(x) = \frac{M_o L}{\mathrm{EI}}\left[\frac{1}{2}(\frac{x}{L})^2 - \frac{1}{6}\right]}$ $\boxed{M(x) = \frac{M_0 x}{L}}$ $\boxed{S(x) = -\frac{M_0}{L}}$
3)		$\boxed{u(x) = \frac{M_0}{4EI}x^2(\frac{x}{L} - 1)}$ $\boxed{u'(x) = \frac{M_0}{4EI}\left(\frac{3x^2}{L} - 2x\right)}$ $\boxed{M(x) = \frac{M_0}{2}\left[\frac{3x}{L} - 1\right]}$ $\boxed{S(x) = -\frac{3M_0}{2L}}$
4)		$\boxed{u(x) = \frac{Px^2(3L - x)}{6\mathrm{EI}}}$ $\boxed{u'(x) = \frac{Px(2L - x)}{2\mathrm{EI}}}$ $\boxed{M(x) = P(L - x)}$ $\boxed{S(x) = P}$
5)		$\boxed{u(x) = \frac{M_0}{2\mathrm{EI}}x^2}$ $\boxed{u'(x) = \frac{M_0}{\mathrm{EI}}x}$ $\boxed{M(x) = M_0}$ $\boxed{S(x) = 0}$

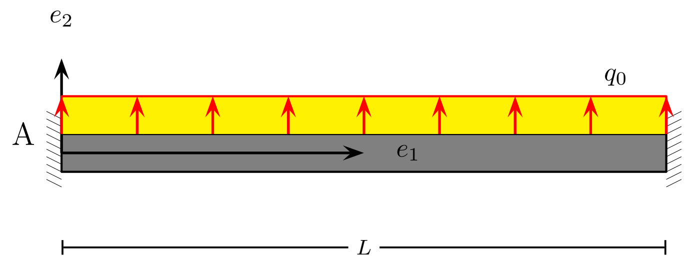


Figure 6

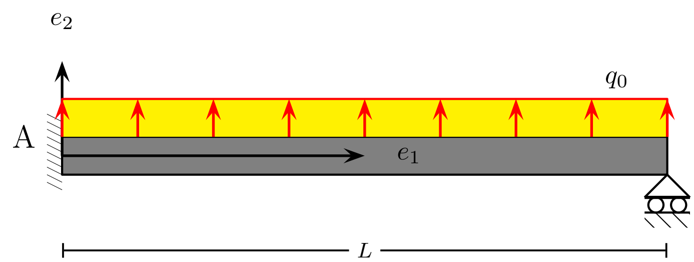


Figure 7

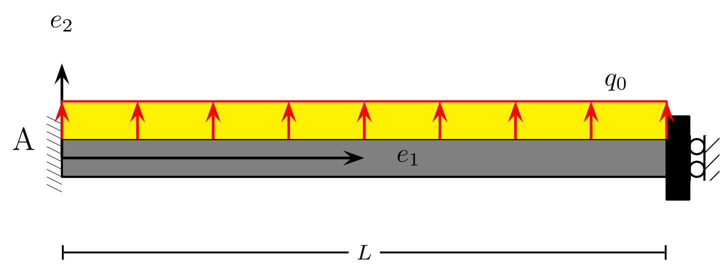


Figure 8

○ **Problems M-12.3** [30 points]

Consider the beams shown in Figures 6-8. The beams have a constant Young's Modulus E, moment of inertia I, width b, and height h. Obtain the following:

3.1 (10 points) The beam deflection distribution $u(x)$

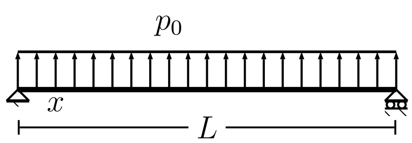
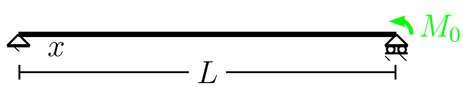
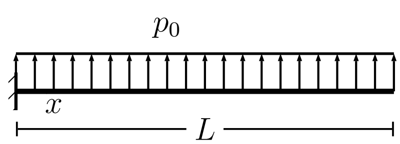
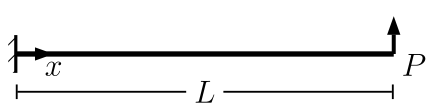
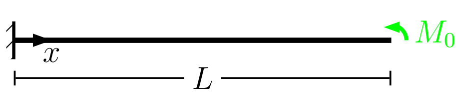
3.2 (10 points) The internal bending moment distribution $M(x)$

3.3 (10 points) The internal shear force distribution $S(x)$

First, complete the above analysis by integration of the governing equations for beam theory.

Next, solve this problem through the use of linear superposition. A number of potentially useful solutions to statically-determinate problems are provided. Explain the process and the solutions you chose to combine for each beam.

Table 1: **solutions to statically-determinate beam problems**

1)		$u(x) = \dfrac{p_0\left(L^3x - 2Lx^3 + x^4\right)}{24EI}$ $u'(x) = \dfrac{p_0\left(L^3 - 6Lx^2 + 4x^3\right)}{24EI}$ $M(x) = \dfrac{1}{2}\left(p_0x^2 - Lp_0x\right)$ $S(x) = \dfrac{1}{2}p_0(L - 2x)$
2)		$u(x) = \dfrac{M_0L^2}{6\text{EI}}(\dfrac{x}{L})\left[(\dfrac{x}{L})^2 - 1\right]$ $u'(x) = \dfrac{M_oL}{\text{EI}}\left[\dfrac{1}{2}(\dfrac{x}{L})^2 - \dfrac{1}{6}\right]$ $M(x) = \dfrac{M_0x}{L}$ $S(x) = -\dfrac{M_0}{L}$
3)		$u(x) = \dfrac{p_0x^2\left(6L^2 - 4Lx + x^2\right)}{24\text{EI}}$ $u'(x) = \dfrac{p_0x\left(3L^2 - 3Lx + x^2\right)}{6\text{EI}}$ $M(x) = \dfrac{1}{2}p_0(L - x)^2$ $S(x) = p_0(L - x)$
4)		$u(x) = \dfrac{Px^2(3L - x)}{6\text{EI}}$ $u'(x) = \dfrac{Px(2L - x)}{2\text{EI}}$ $M(x) = P(L - x)$ $S(x) = P$
5)		$u(x) = \dfrac{M_0}{2\text{EI}}x^2$ $u'(x) = \dfrac{M_0}{\text{EI}}x$ $M(x) = M_0$ $S(x) = 0$

Solution: Beam 1 Starting with the equation

$$(EIw''(x))'' = q_0$$

Integrating four times to get the general solution.

$$EIu(x) = q_0\frac{x^4}{24} + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D$$

The boundary conditions for this problem are:

$$\begin{aligned} w(0) &= 0 \\ w(L) &= 0 \\ w'(0) &= 0 \\ w'(L) &= 0 \end{aligned}$$

Applying them results in

$$\begin{aligned} A &= \frac{-Lq_0}{2} \\ B &= \frac{L^2 q_0}{12} \\ C &= 0 \\ D &= 0 \end{aligned}$$

Thus the fields become

$$\begin{aligned} u(x) &= \frac{q_0}{EI}\left[\frac{x^4}{24} - \frac{L}{12}x^3 + \frac{L^2}{24}x^2\right] \\ M(x) = EIw''(x) &= q_0\left[\frac{x^2}{2} - \frac{Lx}{2} + \frac{L^2}{12}\right] \\ S(x) = -M'(x) &= q_0\left[\frac{L}{2} - x\right] \end{aligned}$$

Now to use superposition to solve. That solution below uses a combination of the provided solution 3, 4,& 5. There are other possible combinations that would work. Using these three solutions, the key is to ensure that w(L) = 0 and w'(L) = 0. From this, you can determine the concentrated load P in solution 4 and M_0 in solution 5 that accomplishes this.

$$\begin{aligned} w_3(L) + w_4(L) + w_5(L) &= 0 \\ \frac{q_0 L^4}{8EI} + \frac{PL^3}{3EI} + \frac{M_0 L^2}{2EI} &= 0 \end{aligned}$$

$$w_3'(L) + w_4'(L) + w_5'(L) = 0$$
$$\frac{q_0 L^3}{6EI} + \frac{PL^2}{2EI} + \frac{M_0 L}{EI} = 0$$

Solving the two equations for P and M_0 yields:

$$P = \frac{-Lq_0}{2}$$
$$M_0 = \frac{L^2 q_0}{12}$$

Combing combing the three solutions with these values plugged in will yield:

$$u(x) = \frac{q_0}{EI}\left[\frac{x^4}{24} - \frac{L}{12}x^3 + \frac{L^2}{24}x^2\right]$$
$$M(x) = EIw''(x) = q_0\left[\frac{x^2}{2} - \frac{Lx}{2} + \frac{L^2}{12}\right]$$
$$S(x) = -M'(x) = q_0\left[\frac{L}{2} - x\right]$$

which are the same fields that you have previously derived.

Solution: Beam 2 Starting with the equation

$$(EIw''(x))'' = q_0$$

Integrating four times to get the general solution.

$$EIu(x) = q_0\frac{x^4}{24} + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D$$

The boundary conditions for this problem are:

$$w(0) = 0$$
$$w(L) = 0$$
$$w'(0) = 0$$
$$M(L) = 0$$

Applying them results in

$$A = \frac{-5Lq_0}{8}$$
$$B = \frac{L^2 q_0}{8}$$
$$C = 0$$
$$D = 0$$

Thus the fields become

$$u(x) = \frac{q_0}{EI}\left[\frac{x^4}{24} - \frac{5L}{48}x^3 + \frac{L^2}{16}x^2\right]$$
$$M(x) = EIw''(x) = q_0\left[\frac{x^2}{2} - \frac{5L}{8}x + \frac{L^2}{8}\right]$$
$$S(x) = -M'(x) = q_0\left[\frac{5L}{8} - x\right]$$

Now use superposition to solve. That solution below uses a combination of the provided solution 3 & 4. There are other possible combinations that would work. Using these two solutions, the key is to ensure that w(L) = 0. From this, you can determine the concentrated load P in solution 4 that accomplishes this.

$$w(L)_3 + w(L)_4 = 0$$
$$\frac{q_0 L^4}{8EI} + \frac{PL^3}{3EI} = 0$$
$$\rightarrow P = \frac{-3q_0 L}{8}$$

Now, plug in the value for P into solution 4 and combined it with solution 3.

$$u(x) = \frac{p_0 x^2\left(6L^2 - 4Lx + x^2\right)}{24\text{EI}} + \frac{\frac{-3q_0L}{8}x^2(3L - x)}{6EI}$$
$$u(x) = \frac{q_0}{EI}\left[\frac{x^4}{24} - \frac{5L}{48}x^3 + \frac{L^2}{16}x^2\right]$$

This is the same solution you derived by integrating the governing equations. Similarly, Moment and Shear will have the same solutions.

$$M(x) = EIw''(x) = q_0\left[\frac{x^2}{2} - \frac{5L}{8}x + \frac{L^2}{8}\right]$$
$$S(x) = -M'(x) = q_0\left[\frac{5L}{8} - x\right]$$

Solution: Beam 3 Starting with the equation

$$(EIw''(x))'' = q_0$$

Integrating four times to get the general solution.

$$EIu(x) = q_0\frac{x^4}{24} + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D$$

The boundary conditions for this problem are:

$$\begin{aligned} w(0) &= 0 \\ w'(0) &= 0 \\ w'(L) &= 0 \\ S(L) &= 0 \end{aligned}$$

Applying the results in

$$\begin{aligned} A &= -Lq_0 \\ B &= \frac{L^2 q_0}{3} \\ C &= 0 \\ D &= 0 \end{aligned}$$

Thus the fields become

$$\begin{aligned} u(x) &= \frac{q_0}{EI}\left[\frac{x^4}{24} - \frac{L}{6}x^3 + \frac{L^2}{6}x^2\right] \\ M(x) &= EIw''(x) = q_0\left[\frac{x^2}{2} - Lx + \frac{L}{3}\right] \\ S(x) &= -M(x) = q_0\left[L - x\right] \end{aligned}$$

Now to use superposition to solve. That solution below uses a combination of the provided solution 3 & 5. There are other possible combinations that would work. Using these two solutions, the key is to ensure that $w'(\mathrm{L}) = 0$. From this, you can determine the concentrated Moment in solution 5 that accomplishes this.

$$\begin{aligned} w'(L)_3 + w'(L)_5 &= 0 \\ \frac{q_0 L^3}{6EI} + \frac{M_0 L}{EI} &= 0 \\ \rightarrow M_0 &= \frac{-q_0 L^2}{6} \end{aligned}$$

Plugging the value of M_0 into the solution for beam 5 and combining it with the solution for beam 3 will yield the following fields, which are the same as you have previously computed:

$$u(x) = \frac{q_0}{EI}\left[\frac{x^4}{24} - \frac{L}{6}x^3 + \frac{L^2}{6}x^2\right]$$
$$M(x) = EIw''(x) = q_0\left[\frac{x^2}{2} - Lx + \frac{L}{3}\right]$$
$$S(x) = -M(x) = q_0\left[L - x\right]$$

○ **Problems M-12.4** [20 points]
Buckling vs yielding

4.1 (20 points) A column has a length L and a constant rectangular cross-section of dimensions $a = 0.1\,\mathrm{m}$ in the x_2-direction and $b = 0.2\,\mathrm{m}$ in the x_3-direction. It is made of a material with $E = 70\,\mathrm{GPa}$ and $\sigma_y = 100\,\mathrm{MPa}$. The column is clamped at $x_1 = 0$. At $x_1 = L$, it is constrained differently in the x_2- and x_3-directions and loaded by a force P as shown in Figure 9. Specifically, in the x_2 plane, the deflection and its derivative are constrained to be zero, while in the x_3 plane both are unconstrained. Compute the buckling load in the second case of buckling in the x_3 plane and compare your results with buckling failure in the other plane and the possibility of column failure occuring due to yielding. An analysis of the x_2 plane is included below for convenience.

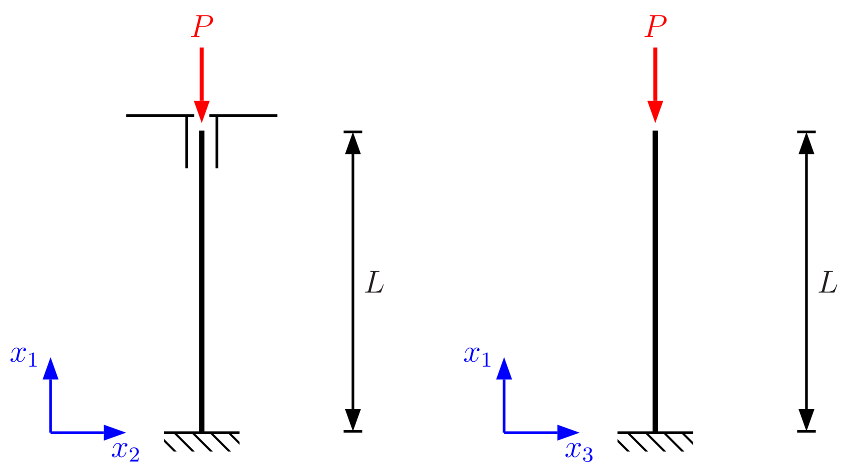


Figure 9: Column constrained differently in the x_2- and x_3-directions at $x_1 = L$.

Determine the maximum value for the length L of the column to guarantee that it will not fail by buckling.

The corresponding fourth-order ODE for the deflection $\bar{u}_2(x_1)$ is

$$EI_{33}\bar{u}_2''''(x_1) + P\bar{u}_2''(x_1) \;=\; p_2(x_1) \;=\; 0.$$

For buckling in this direction, the boundary conditions are:

$$\begin{aligned}\bar{u}_2(0) &= 0\\ \bar{u}_2'(0) &= 0\\ \bar{u}_2(L) &= 0\\ \bar{u}_2'(L) &= 0\end{aligned}$$

Now, apply these boundary conditions and determine the condition on the load P for which the beam can be in equilibrium in a deformed configuration, (i.e. we have a non-trivial solution $\bar{u}_2(x_1) \neq 0$). Also, let's define $k = \sqrt{\frac{P}{EI_{33}}}$.

For $\bar{u}_2(x_1 = 0) = 0$:

$$\begin{aligned} \bar{u}_2(x_1 = 0) &= A\sin(0) + B\cos(0) + 0 + D \\ &= B + D = 0 \end{aligned}$$

For $\bar{u}_2'(x_1 = 0) = 0$:

$$\begin{aligned} \bar{u}_2'(x_1 = 0) &= kA\cos(0) - kB\sin(0) + C \\ &= kA + C = 0 \end{aligned}$$

For $\bar{u}_2(x_1 = L) = 0$:

$$\bar{u}_2'''(x_1 = L) = A\cos(kL) + B\sin(kL) + CL + D = 0$$

For $\bar{u}_2'(x_1 = L) = 0$:

$$\bar{u}_2'(x_1 = L) = kA\cos(kL) - kB\sin(kL) + C = 0$$

$$\left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right\} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ \sin(kL) & \cos(kL) & L & 1 \\ k\cos(kL) & -k\sin(kL) & 1 & 0 \end{bmatrix} \left\{ \begin{array}{c} A \\ B \\ C \\ D \end{array} \right\}$$

For a non-trivial solution (i.e. $\bar{u}_2(x_1) \neq 0$),

$$\left\{ \begin{array}{c} A \\ B \\ C \\ D \end{array} \right\} \neq \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right\},$$

which requires the matrix to be singular, i.e. its determinant must vanish. Let's call this matrix $\mathbf{H}$.

$$\det(\mathbf{H}) = 0 = 2k - 2k\cos(kL) - k^2 L \sin(kL)$$

For the non-trivial case, we must look at the loading conditions that cause this determinant to be zero as is done in the course lecture slides. For the second case of buckling in the other plane, the proceedure is similar but there are different boundary conditions on the position $x_1 = L$. In particular, in the x_3 plane there is nothing touching the column at $x_1 = L$ so there cannot be any moments or shear at the end.

As we have seen in the course the smallest buckling load is given by

$$P_{cr} = \frac{4\pi^2 EI}{L^2}$$

Here the inertia is with respect to the plane of buckling. Perform a similar analysis for the other plane, then compare the critical buckling loads with the analysis given here and with the conditions for failure due to yielding.

Solution:

The corresponding fourth-order ODE for the deflection $\bar{u}_3(x_1)$ is

$$EI_{22}\bar{u}_3''''(x_1) + P\bar{u}_3''(x_1) \;=\; p_3(x_1) \;=\; 0.$$

For buckling in this direction, the applicable boundary conditions are:

$$\begin{aligned}
\bar{u}_3(0) &= 0 \\
\bar{u}_3'(0) &= 0 \\
\bar{u}_3''(L) &= 0 \\
S_3(L) &= -EI_{22}\bar{u}_3'''(L) - P\bar{u}_3'(L) \\
&= 0 \\
&\Rightarrow EI_{22}\bar{u}_3'''(L) + P\bar{u}_3'(L) = 0
\end{aligned}$$

Now apply these boundary conditions and determine the condition on the load P for which the beam can be in equilibrium in a deformed configuration, (i.e. we have a non-trivial solution $\bar{u}_3(x_1) \neq 0$). Also, let's define $k = \sqrt{\frac{P}{EI_{22}}}$.

For $\bar{u}_3(x_1 = 0) = 0$:

$$\begin{aligned}
\bar{u}_3(x_1 = 0) &= A\sin(0) + B\cos(0) + 0 + D \\
&= B + D = 0
\end{aligned}$$

For $\bar{u}_3'(x_1 = 0) = 0$:

$$\begin{aligned} \bar{u}_3'(x_1 = 0) &= kA\cos(0) - kB\sin(0) + C \\ &= kA + C = 0 \end{aligned}$$

For $\bar{u}_3''(x_1 = L) = 0$:

$$\begin{aligned} \bar{u}_3''(x_1 = L) &= -k^2 A\sin(kL) - k^2 B\cos(kL) \\ &= 0 \end{aligned}$$

For $EI_{22}\bar{u}_3'''(x_1 = L) + P\bar{u}_3'(x_1 = L) = 0$:

$$\begin{aligned} &EI_{22}(-k^3 A\cos(kL) + k^3 B\sin(kL)) + P(kA\cos(kL) - kB\sin(kL) + C) \\ &= PC \\ &= 0 \end{aligned}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ -k^2\sin(kL) & -k^2\cos(kL) & 0 & 0 \\ 0 & 0 & P & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix}$$

For a non-trivial solution $\bar{u}_3(x_1)$,

$$\begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} \neq \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

,
which requires the matrix to be singular, i.e. its determinant must vanish. Let's call this matrix **H**.

$$\det(\mathbf{H}) = 0 = k^3 P\cos(kL)$$

For the non-trivial case, we must look at:

$$\begin{aligned} \cos(kL) &= 0 \\ kL &= \frac{\pi}{2} + (n-1)\pi \quad \text{for } n = 1, 2, \ldots \\ P_{cr} &= \pi^2 \left(\frac{1}{2} + (n-1)\right)^2 \frac{EI_{22}}{L^2} \end{aligned}$$

Looking at the critical load (here, $n = 1$ corresponds to the smallest non-trivial critical load)

$$P_{cr} = \frac{\pi^2}{4}\frac{EI_{22}}{L^2} \tag{2}$$

where I_{22} is defined as:

$$I_{22} = \frac{ab^3}{12} = 6.67 \times 10^{-5}\ \mathrm{m}^4$$

Thus, we can now calculate the maximum length of the column that guarantees yielding before buckling to be:

$$\begin{aligned} \sigma_y &\leq \frac{\frac{\pi^2}{4}\frac{EI_{22}}{L^2}}{ba} \\ \Rightarrow\ L &\leq \sqrt{\frac{\pi^2 EI_{22}}{4\sigma_y ba}} \\ \Rightarrow\ L_{\max} &\approx 2.4\,\mathrm{m} \end{aligned}$$

The buckling load in the other plane has a factor of 16 larger buckling load due to the differing boundary conditions, but it also has a four times smaller inertia since in that plane the width and height of the cross section are switched. Thus we can see that the x_3 plane will be the buckling plane. Note that even though the column had a smaller inertia in the x_2 plane that plane is stable with respect to small perturbations because the boundary conditions provide additional stiffness. Also note that the column will fail by yielding if it is made short enough.