16.001 - Materials & Structures
Problem Set #13

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Question	Points
1	34
2	40
3	35
Total:	109

○ **Problem M-13.1**

Beam-column subject to a uniformly-distributed transverse load and a constant compressive force

Consider the beam-column (length l, constant Young's modulus E, and constant second moment of area I pertaining to bending about the x_3-axis) shown in Figure 1 loaded simultaneously by an axial load P and a uniform transverse load $p_2(x_1) = q_0$.

1.1 (2 points) Write the governing equation for this problem based on class discussions.

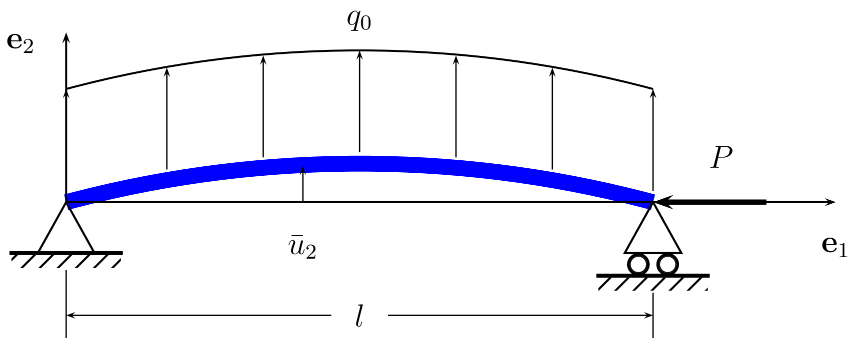


Figure 1: Beam column.

1.2 (2 points) Write the general solution to this problem as the sum of the solution to the homogeneous equation ($q_0 = 0$) given in class and a *particular solution* that satisfies the equation for the load given.

1.3 (5 points) Write down the appropriate boundary conditions for this problem.

1.4 (10 points) Find the unknown constants of the solution by solving the system of algebraic equations resulting from the applicaton of the boundary conditions.

1.5 (5 points) Determine the deflection $\bar{u}_2(x_1)$ by replacing the constants in the general solution.

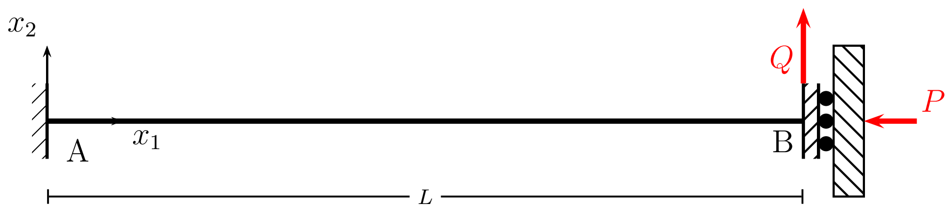
1.6 (5 points) Obtain the moment distribution along the axis of the beam.

1.7 (5 points) Evaluate the bending moment $M(l/2)/M_{\text{primary}}$ at the center, where $M_{\text{primary}} = -q_0 l^2/8$ is the primary bending moment at the center, for axial load ratios $c = P/P_{\text{cr}} = 0.1$, 0.5, 0.8, where $P_{\text{cr}} = \pi^2 EI/l^2$ is the buckling load. Comment on the asymptotic behavior as $c \to 1$.

○ **Problem M-13.2**

Effect of compressive load on fixed beam subject to lateral deflection at one end

Consider the beam of length L and cross-sectional stiffness EI shown in the Figure. A transverse force Q is initially applied at the right end. We are interested in analyzing the effect of an applied compressive load P at the end of the beam in addition to the loading produced by the initial lateral transverse force Q.



2.1 (5 points) Write down the governing equations for the case in which equilibrium is stated in the deformed configuration as well as the boundary conditions.

2.2 (2 points) Is this a homogeneous or inhomogeneous problem? Why?

2.3 (3 points) The solution to the general homogeneous equation $u'''' + k^2 u'' = 0$ is:

$$u(x) = A\sin(kx) + B\cos(kx) + Cx + D$$

Specialize it to our problem by proving that $k = \sqrt{\frac{P}{EI}}$. Can this solution be used for our problem? Why/why not?

2.4 (10 points) Use the boundary conditions of the problem to obtain algebraic equations for the constants A, B, C and D in the solution. Explain why you can solve the equations for the constants in this case. Go ahead and obtain the values for the constants, the deflection $u(x)$ and the moment distribution $M(x)$

2.5 (5 points) Evaluate the moment at $x = L$. Find the limit of $M(L)$ when $k \to 0$
Hint: Use L'Hôpital as many times as necessary to resolve the indeterminacy.

2.6 (5 points) From the solution from beam theory derived in the previous problem, we find that the moment and the shear at $x = L$ are related by $M(L) = -QL/2$. Show that the beam-column solution matches this limit when the compressive load is very small, i.e. $P \to 0$. (*Hint: use* $k^2 = P/EI$)

2.7 (5 points) For what value of the compressive load does $M(L)$ (and the rest of the solution) grow unboundedly?(*Hint: use* $k^2 = P/EI$*)* What does this value represent?

2.8 (5 points) Show that when the compressive force $P = \frac{1}{4}P_{cr}$, the moment at L predicted by beam-column theory is a factor of $\frac{4}{\pi} \sim 27\%$ larger than that predicted by beam theory.

Problem M-13.3

Response to compressive loads of a simply-supported beam subject to eccentric load

Consider the uniform simply-supported beam of length L and cross-sectional stiffness EI shown in Figure 3. The beam is loaded with an eccentric compressive load P at the right end. We are interested in analyzing the influence of the eccentricity e in the bending vs. buckling response of the beam. The deflection of the axis of the beam will be described by the function $u_2(x_1)$. For the sake of simplicity, we will rename the variables $x_1 \rightarrow x$, and $u_2 \rightarrow u$.

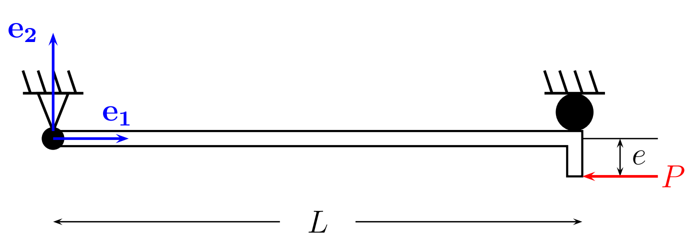


Figure 2: Simply supported beam subject to eccentric compressive load P

3.1 (5 points) Write down the governing equations for the case in which equilibrium is stated in the deformed configuration as well as the boundary conditions.

3.2 (2 points) Is this a homogeneous or inhomogeneous problem? Why?

3.3 (3 points) The solution to the general homogeneous equation $u'''' + k^2 u'' = 0$ is:

$$u(x) = A\sin(kx) + B\cos(kx) + Cx + D$$

Specialize it to our problem by proving that $k = \sqrt{\frac{P}{EI}}$. Can this solution be used for our problem? Why/why not?

3.4 (10 points) Use the boundary conditions of the problem to obtain algebraic equations for the constants A, B, C and D in the solution. Explain why you can solve the equations for the constants in this case. Go ahead and obtain the values for the constants and show that the solution is:

$$u(x) = e\left[\frac{\sin(kx)}{\sin(kL)} - \frac{x}{L}\right]$$

$$M(x) = -Pe\frac{\sin(kx)}{\sin(kL)}$$

where $k = \sqrt{\frac{P}{EI}}$

3.5 (5 points) The solution from beam theory is:

$$u(x) = \frac{PeL^2}{6EI}\left(\frac{x}{L}\right)\left[1 - \left(\frac{x}{L}\right)^2\right]$$

$$M(x) = -Pe\frac{x}{L}$$

which is clearly different from that of beam-column theory above.

Show that the ratio of the deflections and the bending moments half-way through the span of the beam (i.e. at $x = L/2$) predicted by the two theories are respectively given by the expressions:

$$r_\delta(k) = \frac{\delta^{\text{buckling}}}{\delta^{\text{beam theory}}} = \frac{u^{\text{buckling}}(L/2)}{u^{\text{beam theory}}(L/2)} = \frac{8}{k^2L^2}\left[\sec\left(\frac{kL}{2}\right) - 1\right]$$

$$r_M(k) = \frac{M^{\text{buckling}}(L/2)}{M^{\text{beam theory}}(L/2)} = \sec\left(\frac{kL}{2}\right)$$

3.6 (5 points) Use the expression for $r_M(k)$ to interpret what happens with the moment at $x = L/2$ as the compressive load increases? In particular, what happens with that moment when $\frac{kL}{2} \to \frac{\pi}{2}$?

3.7 (5 points) Challenge: Use the expression for $r_\delta(k)$ to show that when the beam is very stiff (i.e $EI/L^2 \gg P$), beam-column theory and beam theory give the same deflection. *Hints: Recall the definition of k from above, Use L'Hôpital as many times as needed to compute the appropriate limit of* $r_\delta(k)$. *Also* $\sec'(x) = \sec(x)\tan(x), \tan'(x) = \sec^2(x)$