16.001 - Materials & Structures
Problem Set #13

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Question	Points
1	34
2	40
3	35
Total:	109

○ **Problem M-13.1**

Beam-column subject to a uniformly-distributed transverse load and a constant compressive force

Mathematica Notebook Consider the beam-column (length l, constant Young's modulus E, and constant second moment of area I pertaining to bending about the x_3-axis) shown in Figure 1 loaded simultaneously by an axial load P and a uniform transverse load $p_2(x_1) = q_0$.

1.1 (2 points) Write the governing equation for this problem based on class discussions.

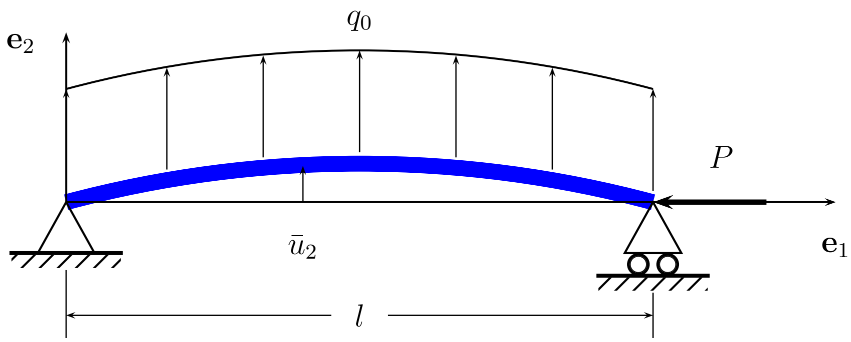


Figure 1: Beam column.

Solution: From class discussions:

$$(EI\bar{u}_2'')'' + P\bar{u}_2'' = q_0 \tag{1}$$

1.2 (2 points) Write the general solution to this problem as the sum of the solution to the homogeneous equation ($q_0 = 0$) given in class and a *particular solution* that satisfies the equation for the load given.

Solution: The homogeneous solutions was:

$$\bar{u}_2^h(x_1) \;=\; A\sin(\sqrt{\frac{P}{EI}}x_1) + B\cos(\sqrt{\frac{P}{EI}}x_1) + Cx_1 + D$$

The particular solution sought is:

$$\bar{u}_2^p(x_1) = \frac{q_0}{2P}x_1^2$$

1.3 (5 points) Write down the appropriate boundary conditions for this problem.

Solution: As before, for a simply supported beam, we obtain:

$$\bar{u}_2(0) \;=\; 0 \tag{2}$$
$$M(0) = EI\bar{u}_2''(0) \;=\; 0 \tag{3}$$
$$\bar{u}_2(l) \;=\; 0 \tag{4}$$
$$M(l) = EI\bar{u}_2''(l) \;=\; 0 \tag{5}$$

1.4 (10 points) Find the unknown constants of the solution by solving the system of algebraic equations resulting from the applicaton of the boundary conditions.

Solution:

$$(3) \quad \Rightarrow \quad -B + \frac{q_0 EI}{P^2} = 0 \quad \Rightarrow \quad B = \frac{q_0 EI}{P^2}$$

$$(2) \quad \Rightarrow \quad B + D = 0 \quad \Rightarrow \quad D = -\frac{q_0 EI}{P^2}$$

$$(5) \quad \Rightarrow \quad A \sin(\sqrt{\frac{P}{EI}} l) + B \cos(\sqrt{\frac{P}{EI}} l) - \frac{q_0 EI}{P^2} = 0$$

$$\Rightarrow \quad A = \frac{q_0 EI}{P^2} \frac{\left[1 - \cos(\sqrt{\frac{P}{EI}} l)\right]}{\sin(\sqrt{\frac{P}{EI}} l)}$$

$$(4) \quad \Rightarrow \quad A \sin(\sqrt{\frac{P}{EI}} l) + B \cos(\sqrt{\frac{P}{EI}} l) + Cl + D + \frac{q_0}{2P} l^2 = 0$$

$$\Rightarrow \quad C = -\frac{q_0}{2P} l$$

1.5 (5 points) Determine the deflection $\bar{u}_2(x_1)$ by replacing the constants in the general solution.

Solution:

$$\begin{aligned}
\bar{u}_2(x_1) &= \frac{q_0 EI}{P^2} \frac{[1 - \cos(\sqrt{\frac{P}{EI}} l)]}{\sin(\sqrt{\frac{P}{EI}} l)} \sin(\sqrt{\frac{P}{EI}} x_1) + \frac{q_0 EI}{P^2} \cos(\sqrt{\frac{P}{EI}} x_1) \\
&\quad - \frac{q_0}{2P} l x_1 - \frac{q_0 EI}{P^2} + \frac{q_0}{2P} x_1^2 \\
&= \frac{q_0 EI}{P^2} \left(\left(1 - \cos(\sqrt{\frac{P}{EI}} l)\right) \frac{\sin(\sqrt{\frac{P}{EI}} x_1)}{\sin(\sqrt{\frac{P}{EI}} l)} - \left(1 - \cos(\sqrt{\frac{P}{EI}} x_1)\right) \right) \\
&\quad - \frac{q_0}{2P} x_1 (l - x_1)
\end{aligned}$$

1.6 (5 points) Obtain the moment distribution along the axis of the beam.

Solution:

$$M(x_1) = \frac{q_0 EI}{P} \left[\left(\cos(\sqrt{\frac{P}{EI}} l) - 1\right) \frac{\sin(\sqrt{\frac{P}{EI}} x_1)}{\sin(\sqrt{\frac{P}{EI}} l)} - \cos(\sqrt{\frac{P}{EI}} x_1) + 1 \right]$$

1.7 (5 points) Evaluate the bending moment $M(l/2)/M_{\text{primary}}$ at the center, where $M_{\text{primary}} = -q_0 l^2/8$ is the primary bending moment at the center, for axial load ratios $c = P/P_{\text{cr}} = 0.1, 0.5, 0.8$, where $P_{\text{cr}} = \pi^2 EI/l^2$ is the buckling load. Comment on the asymptotic behavior as $c \to 1$.

Solution: The ratio $M(x_1)$ over M_{primary} can be written as follows:

$$\frac{M(x_1)}{M_{\text{primary}}} = -\frac{8}{c\pi^2}\Big[\big(\cos(\pi\sqrt{c}) - 1\big)\frac{\sin(\frac{\pi x_1}{l}\sqrt{c})}{\sin(\pi\sqrt{c})} - \cos(\frac{\pi x_1}{l}\sqrt{c}) + 1\Big]$$

At the center of the beam, it becomes:

$$\frac{M(l/2)}{M_{\text{primary}}} = -\frac{8}{c\pi^2}\Big[\big(\cos(\pi\sqrt{c}) - 1\big)\frac{\sin(\frac{\pi}{2}\sqrt{c})}{\sin(\pi\sqrt{c})} - \cos(\frac{\pi}{2}\sqrt{c}) + 1\Big]$$

The values of this function for $c = 0.1$, 0.5 and 0.8 are tabulated below:

c	0.1	0.5	0.8
$\lvert M(l/2)/M_{\text{primary}}\rvert$	1.11	2.03	5.12

The asymptotic behavior the ratio of moments as a function of the ratio of the applied load by the critical load is depicted in Figure 2.

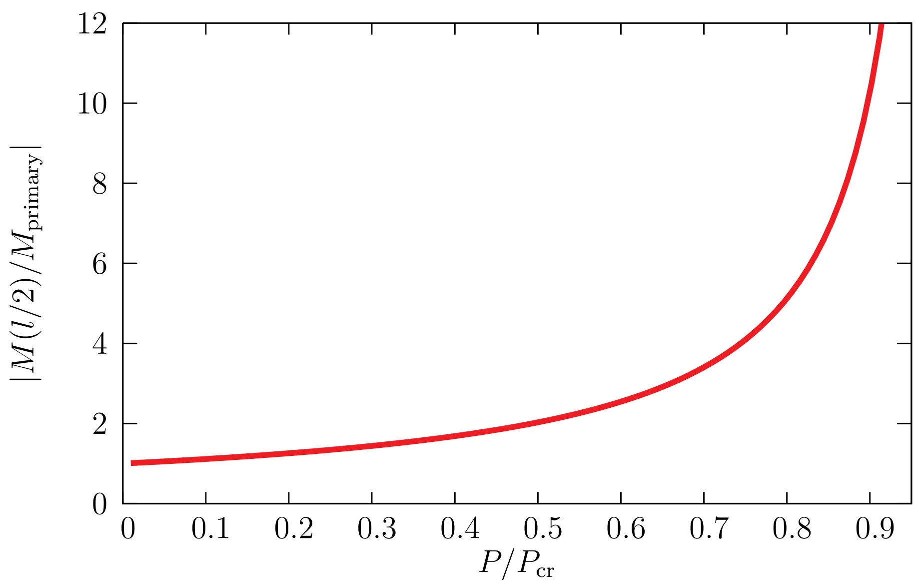
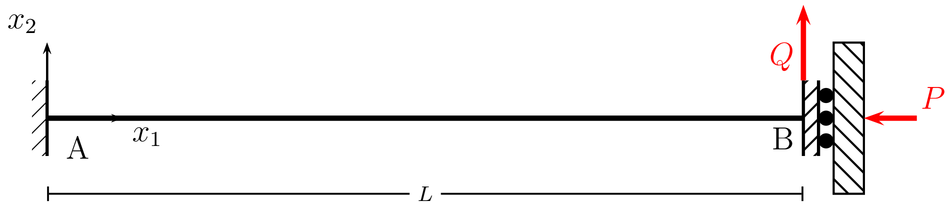


Figure 2: Graph of the ratio of the moment $|M(l/2)|$ normalized by the primary moment $|M_{\text{primary}}| = q_0 l^2/8$ at the center of the beam as a function of the ratio between the applied load P and the critical load P_{cr}. The ratio of moments $|M(l/2)/M_{\text{primary}}| \to \infty$ as the load tends to the critical load.

○ **Problem M-13.2**

Effect of compressive load on fixed beam subject to lateral deflection at one end
Consider the beam of length L and cross-sectional stiffness EI shown in the Figure. A transverse force Q is initially applied at the right end. We are interested in analyzing the effect of an applied compressive load P at the end of the beam in addition to the loading produced by the initial lateral transverse force Q.



2.1 (5 points) Write down the governing equations for the case in which equilibrium is stated in the deformed configuration as well as the boundary conditions.

Solution: Governing equation and boundary conditions:

$$S' + M = 0$$
$$M' + Pu' + S = 0$$
$$M = EIu''$$
$$EIu'''' + Pu'' = 0$$
$$u(0) = u'(0) = 0$$
$$u'(L) = 0$$
$$S(L) = -EIu'''(L) - \cancelto{0}{Pu'(L)} = Q$$

Note that the $Pu'(L)$ term in the shear boundary condition at L should in principle be included (this actually has the benefit that it simplifies the algebraic equation). But since $u'[L] = 0$, it can also be omitted.

2.2 (2 points) Is this a homogeneous or inhomogeneous problem? Why?

Solution: inhomogeneous from the boundary conditions, the equation itself is homogeneous.

2.3 (3 points) The solution to the general homogeneous equation $u'''' + k^2u'' = 0$ is:

$$u(x) = A\sin(kx) + B\cos(kx) + Cx + D$$

Specialize it to our problem by proving that $k = \sqrt{\frac{P}{EI}}$. Can this solution be used for our problem? Why/why not?

Solution: Our governing equation can be rewritten as: $u'''' + \frac{P}{EI}u'' = 0$. By analogy, $k^2 = \frac{P}{EI}$.
Yes, it can be used, since our equation is homogeneous. Our inhomogeneity comes from the boundary conditions.

2.4 (10 points) Use the boundary conditions of the problem to obtain algebraic equations for the constants A, B, C and D in the solution. Explain why you can solve the equations for the constants in this case. Go ahead and obtain the values for the constants, the deflection $u(x)$ and the moment distribution $M(x)$

Solution: Can solve because the system is inhomogeneous: as long as the matrix of coefficients is invertible, I can obtain the constants.

$$\begin{aligned}
0 = u(0) &= A\sin(k\times 0) + B\cos(k\times 0) + C\times 0 + D \\
&= \boxed{B + D = 0} \\
0 = u'(0) &= kA\cos(k\times 0) - kB\sin(k\times 0) + C,\ \boxed{kA + C = 0} \\
0 = u'(L) &= \boxed{kA\cos(k\times L) - kB\sin(k\times L) + C = 0} \\
u'''(L) = -\frac{Q}{\underbrace{(EI)}_{P/k^2}} &= -\frac{Qk^2}{P} = k^3(B\sin(kL) - A\cos(kL)
\end{aligned}$$

$$\boxed{\frac{Q}{Pk} = A\cos(kL) - B\sin(kL)}$$

Replace the 2nd in the 3rd (obtain an equation in A, B, solve with the 4th to obtain:

$$\boxed{A = \frac{Q}{Pk}}$$

$$\boxed{B = \frac{(\cos(kL) - 1)}{\sin(kL)}\frac{Q}{Pk} = -D}$$

$$\boxed{C = -\frac{Q}{P}}$$

Then, the solution is:

$$\boxed{u(x) = \frac{Q}{Pk}\left[\sin(kx) - kx + \frac{(\cos(kL) - 1)}{\sin(kL)}(\cos kx - 1)\right]}$$

$$M(x) = EIu''(x) = EI\frac{Q}{Pk}(-k^2)\left[\sin(kx) + \frac{(\cos(kL)-1)}{\sin(kL)}\cos kx\right]$$

$$\boxed{M(x) = -\frac{Q}{k}\left[\sin(kx) + \frac{(\cos(kL)-1)}{\sin(kL)}\cos(kx)\right]}$$

2.5 (5 points) Evaluate the moment at $x = L$. Find the limit of $M(L)$ when $k \to 0$
Hint: Use L'Hôpital as many times as necessary to resolve the indeterminacy.

Solution:

$$M(L) = -\frac{Q}{k}\left[\sin(kL) + \frac{(\cos(kL) - 1)}{\sin(kL)}\cos(kL)\right]$$

which can be written as:

$$M(L) = -\frac{Q}{k}\left[\frac{(1 - \cos(kL))}{\sin(kL)}\right]$$

at $k = 0$ there is a $0/0$ indeterminacy. Apply L'Hôpital:

$$\begin{aligned}\lim_{k\to 0} M(L) &= \lim_{k\to 0} -\frac{Q}{k}\frac{(1 - \cos(kL))}{\sin(kL)} \\ &= \lim_{k\to 0} -Q\frac{L\sin(kL)}{kL\cos(kL) + \sin(kL)} \\ &= \lim_{k\to 0} -Q\frac{L^2\cos(kL)}{-kL^2\sin(kL) + 2L\cos(kL)} = -\frac{QL}{2}\end{aligned}$$

2.6 (5 points) From the solution from beam theory derived in the previous problem, we find that the moment and the shear at $x = L$ are related by $M(L) = -QL/2$. Show that the beam-column solution matches this limit when the compressive load is very small, i.e. $P \to 0$. (*Hint: use* $k^2 = P/EI$)

Solution: In the first problem, we computed the shear and moment at $x = L$ in terms of the applied δ. We can eliminate δ and obtain the moment as a function of the shear at that point. We had:

$$S(L) = 12\delta\frac{EI}{L^3}, \to \delta = \frac{QL^3}{12EI}$$

$$M(L) = -6\frac{EI}{L^2}\delta = -6\frac{\cancel{EI}}{\cancel{L^2}}\frac{QL^{\cancel{3}}}{12\cancel{EI}} = -\frac{QL}{2}$$

i.e. the two values coincide

2.7 (5 points) For what value of the compressive load does $M(L)$ (and the rest of the solution) grow unboundedly?(*Hint: use* $k^2 = P/EI$) What does this value represent?

Solution: Clearly, when $\sin(kL) = 0$ for non-zero k, i.e. $k_{cr}L = \pi$, leading to the critical load: $P_{cr} = \frac{\pi^2 EI}{L^2}$, the same as a simply-supported beam.

2.8 (5 points) Show that when the compressive force $P = \frac{1}{4}P_{cr}$, the moment at L predicted by beam-column theory is a factor of $\frac{4}{\pi} \sim 27\%$ larger than that predicted by beam theory.

Solution:

$$\frac{P}{P_{cr}} = \left(\frac{k}{k_{cr}}\right)^2 = \frac{1}{4}$$
$$\frac{k}{k_{cr}} = \frac{1}{2},\ kL = \frac{1}{2}k_{cr}L = \frac{1}{2}\pi$$

Replace kL in the expression for $M(L)$ and obtain:

$$M(L) = -\frac{Q}{\pi/(2L)}\left(\frac{1-\cos(\pi/2)}{\sin(\pi/2)}\right) = -QL\frac{2}{\pi}$$

Dividing the two predictions:

$$\frac{M^{\text{b.c}}(L)}{M^{\text{beam}}(L)} = \frac{-QL\frac{2}{\pi}}{-\frac{QL}{2}} = \frac{4}{\pi} \sim 1.27$$

○ **Problem M-13.3**

Response to compressive loads of a simply-supported beam subject to eccentric load
Consider the uniform simply-supported beam of length L and cross-sectional stiffness EI shown in Figure 3. The beam is loaded with an eccentric compressive load P at the right end. We are interested in analyzing the influence of the eccentricity e in the bending vs. buckling response of the beam. The deflection of the axis of the beam will be described by the function $u_2(x_1)$. For the sake of simplicity, we will rename the variables $x_1 \to x$, and $u_2 \to u$.

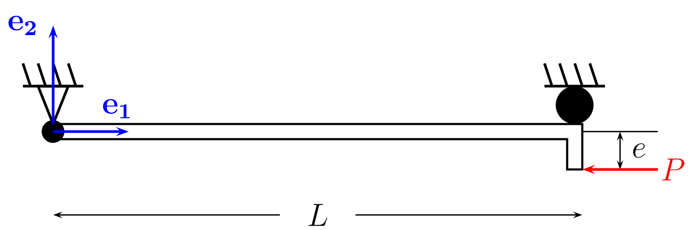


Figure 3: Simply supported beam subject to eccentric compressive load P

3.1 (5 points) Write down the governing equations for the case in which equilibrium is stated in the deformed configuration as well as the boundary conditions.

Solution: Governing equation and boundary conditions:

$$EIu'''' + Pu'' = 0$$
$$u(0) = u(L) = 0$$
$$M(0) = EIu''(0) = 0$$
$$M(L) = EIu''(L) = -Pe$$

3.2 (2 points) Is this a homogeneous or inhomogeneous problem? Why?

Solution: inhomogeneous from the boundary conditions, the equation itself is homogeneous.

3.3 (3 points) The solution to the general homogeneous equation $u'''' + k^2 u'' = 0$ is:

$$u(x) = A\sin(kx) + B\cos(kx) + Cx + D$$

Specialize it to our problem by proving that $k = \sqrt{\frac{P}{EI}}$. Can this solution be used for our problem? Why/why not?

Solution: Our governing equation can be rewritten as: $u'''' + \frac{P}{EI}u'' = 0$. By analogy, $k^2 = \frac{P}{EI}$.
Yes, it can be used, since our equation is homogeneous. Our inhomogeneity comes from the boundary conditions.

3.4 (10 points) Use the boundary conditions of the problem to obtain algebraic equations for the constants A, B, C and D in the solution. Explain why you can solve the equations for the constants in this case. Go ahead and obtain the values for the constants and show that the solution is:

$$u(x) = e\left[\frac{\sin(kx)}{\sin(kL)} - \frac{x}{L}\right]$$

$$M(x) = -Pe\frac{\sin(kx)}{\sin(kL)}$$

where $k = \sqrt{\frac{P}{EI}}$

Solution: Can solve because the system is inhomogeneous: as long as the matrix of coefficients is invertible, I can obtain the constants.

$$\begin{aligned}
0 = u(0) &= A\sin(k\times 0) + B\cos(k\times 0) + C\times 0 + D \\
&= \boxed{B + D = 0} \\
0 = M(0) = EIu''(0) &= -k^2\left(A\sin(k\times 0) + B\cos(k\times 0)\right), \ \boxed{B = 0, D = 0} \\
0 = u(L) &= \boxed{A\sin(kL) + CL = 0} \\
-Pe = M(L) &= EIu''(L) = \boxed{-AP\sin(kL) = -Pe}
\end{aligned}$$

The solution is:

$$\boxed{A = \frac{e}{\sin(kL)}}$$

$$\boxed{C = -\frac{e}{L}}$$

$$\boxed{B = D = 0}$$

Then, the solution is:

$$\boxed{u(x) = e\left[\frac{\sin(kx)}{\sin(kL)} - \left(\frac{x}{L}\right)\right]}$$

$$\boxed{M(x) = EIu''(x) = \underbrace{EI(-k^2)}_{-P}\, e\left[\frac{\sin(kx)}{\sin(kL)}\right]}$$

where $k = \sqrt{\frac{P}{EI}}$

3.5 (5 points) The solution from beam theory is:

$$u(x) = \frac{PeL^2}{6EI}\left(\frac{x}{L}\right)\left[1 - \left(\frac{x}{L}\right)^2\right]$$

$$M(x) = -Pe\frac{x}{L}$$

which is clearly different from that of beam-column theory above.

Show that the ratio of the deflections and the bending moments half-way through the span of the beam (i.e. at $x = L/2$) predicted by the two theories are respectively given by the expressions:

$$r_\delta(k) = \frac{\delta^{\text{buckling}}}{\delta^{\text{beam theory}}} = \frac{u^{\text{buckling}}(L/2)}{u^{\text{beam theory}}(L/2)} = \frac{8}{k^2L^2}\left[\sec\left(\frac{kL}{2}\right) - 1\right]$$

$$r_M(k) = \frac{M^{\text{buckling}}(L/2)}{M^{\text{beam theory}}(L/2)} = \sec\left(\frac{kL}{2}\right)$$

Solution: From the previous question:

$$\delta^{\text{buckling}} = e\left[\frac{\cancel{\sin\left(\frac{kL}{2}\right)}}{\underbrace{2\cancel{\sin\left(\frac{kL}{2}\right)}\cos\left(\frac{kL}{2}\right)}_{\sin kL}} - \frac{1}{2}\right] = \frac{e}{2}\left[\sec\left(\frac{kL}{2}\right) - 1\right]$$

From the solution for beams provided:

$$\delta^{\text{beam theory}} = \frac{e}{6}k^2L^2\frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{ek^2L^2}{16}$$

Then:

$$r_\delta(k) = \frac{8}{k^2L^2}\left[\sec\left(\frac{kL}{2}\right) - 1\right]$$

Similarly, obtain $r_M(k)$

3.6 (5 points) Use the expression for $r_M(k)$ to interpret what happens with the moment at $x = L/2$ as the compressive load increases? In particular, what happens with that moment when $\frac{kL}{2} \to \frac{\pi}{2}$?

Solution: $r_M(P) = \sec\left(\frac{kL}{2}\right) = \sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)$ adopts the following values: $r_M(P \to 0) = 1$, i.e. the bending moment coincides with the value from beam theory. The secant is an increasing function (the moment in the beam-column is increasingly larger than in beam theory) with a vertical asymptote at $\pi/2$, that is, when $k \to \pi/L$, $k^2 = \frac{P}{EI} \to \pi^2/L^2$, or $P \to \frac{\pi^2 EI}{L^2} = P_{cr}$ for this beam. This means that as the critical load for this beam is reached, the moment grows unboundedly. .

3.7 (5 points) Challenge: Use the expression for $r_\delta(k)$ to show that when the beam is very stiff (i.e $EI/L^2 \gg P$), beam-column theory and beam theory give the same deflection. *Hints: Recall the definition of k from above, Use L'Hôpital as many times as needed to compute the appropriate limit of* $r_\delta(k)$. *Also* $\sec'(x) = \sec(x)\tan(x), \tan'(x) = \sec^2(x)$

Solution: $k^2 = \frac{P}{EI}$. When $EI/L^2 \gg P$, $\frac{P}{EI}L^2 = k^2L^2 \ll 1 \;\Rightarrow\; kL \to 0$. $r(k)$ has an indeterminacy of the type $\frac{0}{0}$ at $kL = 0$, but the limit $\lim_{kL\to 0} r(k)$ exists and is given by:

$$\begin{aligned}\lim_{kL\to 0} \frac{8}{k^2L^2}\left[\sec\left(\frac{kL}{2}\right) - 1\right] &= \lim_{kL\to 0} \frac{4}{kL}\left[\frac{1}{2}\sec\left(\frac{kL}{2}\right)\tan\left(\frac{kL}{2}\right)\right] \\ &= \lim_{kL\to 0} 2\left[\frac{1}{2}\sec\left(\frac{kL}{2}\right)\tan^2\left(\frac{kL}{2}\right) + \frac{1}{2}\sec^3\left(\frac{kL}{2}\right)\right] = 1 \qquad (6)\end{aligned}$$

Clearly, $r(k \to 0) = 1$ implies that the two solutions coincide.