F7 – Lecture Notes

1. Elliptical Lift Distribution

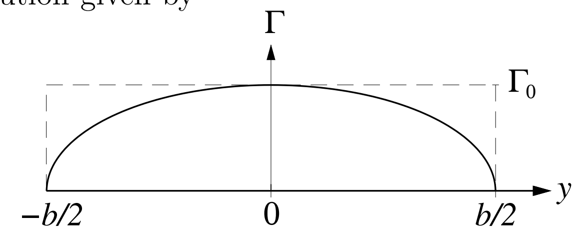
Reading: Anderson 5.3.1

Elliptical Lift Distribution

Definition and lift calculation

Consider an elliptical spanwise circulation distribution given by

$$\Gamma(y) \;=\; \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$



where Γ_0 is the circulation at the wing center at $y = 0$. The overall lift on the wing is the integral of the corresponding lift/span distribution $L'(y) \;=\; \rho V_\infty \Gamma(y)$.

$$L \;=\; \int_{-b/2}^{b/2} L'(y)\, dy \;=\; \int_{-b/2}^{b/2} \rho V_\infty \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}\, dy \;=\; \frac{\pi}{4} \rho V_\infty \Gamma_0 b$$

The integral can be evaluated via integral tables, or by inspection by noting that the area under an ellipse is $\pi/4$ times the area of the enclosing rectangle.

Downwash calculation

Computation of the downwash first requires knowing the trailing vortex sheet strength, which is minus the derivative of the circulation.

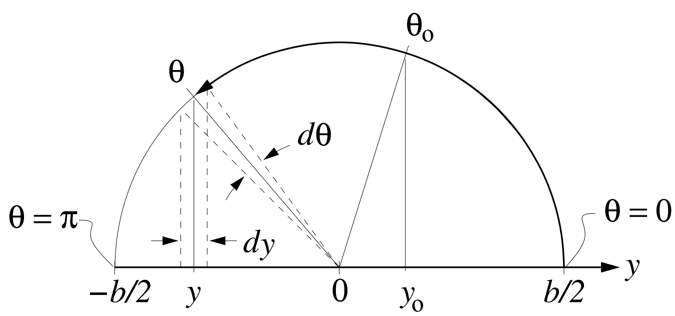
$$\gamma(y) \;=\; -\frac{d\Gamma}{dy} \;=\; \frac{4}{b^2} \frac{y}{\sqrt{1-(2y/b)^2}}$$

The downwash at some location y_o is then

$$w(y_o) \;=\; \frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{\gamma(y)\, dy}{y_o - y} \;=\; \frac{\Gamma_0}{\pi b^2} \int_{-b/2}^{b/2} \frac{y}{\sqrt{1-(2y/b)^2}} \frac{dy}{y_o - y}$$

As in thin airfoil theory, the mathematical problem is considerably simplified by making the trigonometric substitution

$$y_o \;=\; \frac{b}{2}\cos\theta_o$$
$$y \;=\; \frac{b}{2}\cos\theta$$
$$dy \;=\; -\frac{b}{2}\sin\theta\, d\theta$$



The downwash integral then becomes

$$w(\theta_o) \;=\; -\frac{\Gamma_0}{2\pi b} \int_{\pi}^{0} \frac{\cos\theta}{\cos\theta_o - \cos\theta}\, d\theta \;=\; \frac{\Gamma_0}{2\pi b} \int_{0}^{\pi} \frac{\cos\theta}{\cos\theta_o - \cos\theta}\, d\theta$$

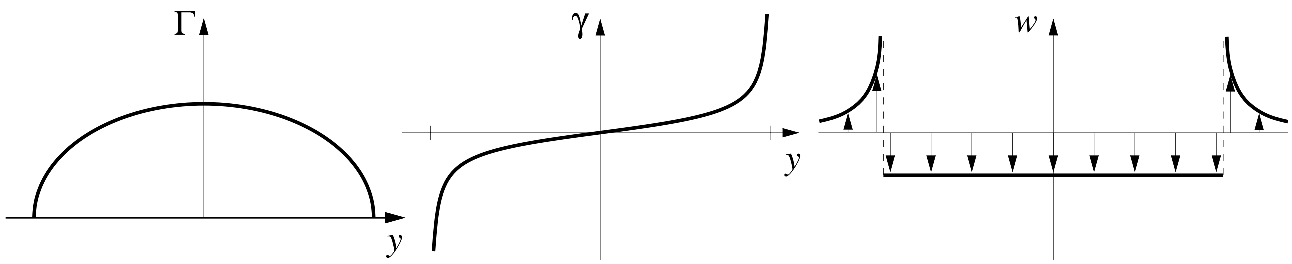
This can be evaluated using the known tabulated integral

$$\int_0^{\pi} \frac{\cos n\theta}{\cos\theta - \cos\theta_o}\, d\theta \;=\; \frac{\pi \,\sin n\theta_o}{\sin\theta_o}$$

Our case corresponds to $n = 1$, and hence

$$w(\theta_o) \;=\; \frac{\Gamma_0}{2\pi b}\left(-\frac{\pi\,\sin\theta_o}{\sin\theta_o}\right) \;=\; -\frac{\Gamma_0}{2b}$$

We have the somewhat surprising result that the downwash is uniform over the span of a wing with an elliptical circulation distribution. There is a sharp upwash just outboard of the tips which rapidly dies off with distance, but this doesn't impact the flow angles seen by the wing itself.



From the earlier lift expression above we have

$$\Gamma_0 \;=\; \frac{4L}{\rho V_\infty b\, \pi}$$

which allows elimination of Γ_0 from the w result to give a somewhat more convenient expression.

$$w \;=\; -\frac{2L}{\rho V_\infty b^2\, \pi}$$

Induced angles

Because w is uniform across the span, the induced angles are uniform as well.

$$\alpha_i \;=\; -\frac{w}{V_\infty} \;=\; \frac{\Gamma_0}{2bV_\infty} \;=\; \frac{L}{\frac{1}{2}\rho V_\infty^2 b^2\, \pi}$$

We can also use the definition of the overall lift coefficient

$$C_L \;\equiv\; \frac{L}{\frac{1}{2}\rho V_\infty^2 S}$$

to obtain yet a third expression for the induced angle.

$$\alpha_i \;=\; \frac{S\, C_L}{b^2\, \pi} \;=\; \frac{C_L}{\pi\, AR}$$

Induced drag

Because α_i for the elliptically-loaded wing is constant along the span, all the lift vectors along the span are tilted by the same amount, making the calculation of induced drag relatively simple.

$$D_i \;=\; \int_{-b/2}^{b/2} L'(y)\,\alpha_i\,dy \;=\; \alpha_i \int_{-b/2}^{b/2} L'(y)\,dy \;=\; \alpha_i\,L$$

Substituting for α_i in terms of the lift itself gives

$$D_i \;=\; \frac{(L/b)^2}{\frac{1}{2}\rho\,V_\infty^2\,\pi}$$

Dividing by $\frac{1}{2}\rho V_\infty^2 S$ gives the equivalent dimensionless relation.

$$C_{Di} \;=\; \frac{C_L^2}{\pi\,AR}$$

The dimensional and dimensionless forms for induced drag above are both useful. Which one would be used in practice depends on the design or analysis application at hand.

Total wing drag

The overall wing drag is equal to the profile drag plus the induced drag.

$$D \;=\; D_p \,+\, D_i$$

$$\text{or} \qquad C_D \;=\; C_{Dp} \,+\, C_{Di}$$

The profile drag coefficient is the chord-weighted average of the local $c_d(y)$.

$$C_{Dp} \;=\; \frac{1}{S}\int_{-b/2}^{b/2} c_d(y)\,c(y)\,dy$$

Here c_d is the 2-D airfoil viscous airfoil drag, and is usually known in the form of a $c_d(c_\ell; Re)$ drag polar from wind tunnel data or from calculations. In general, $c_d(y)$ will vary across the span, although a very common approximation is to simply assume that it's constant, and determined using the overall wing C_L, and the Reynolds number based on the average chord.

$$c_d \;\simeq\; c_d\,(C_L; Re_{\mathrm{avg}}) \qquad , \qquad Re_{\mathrm{avg}} = \frac{V_\infty c_{\mathrm{avg}}}{\nu}$$

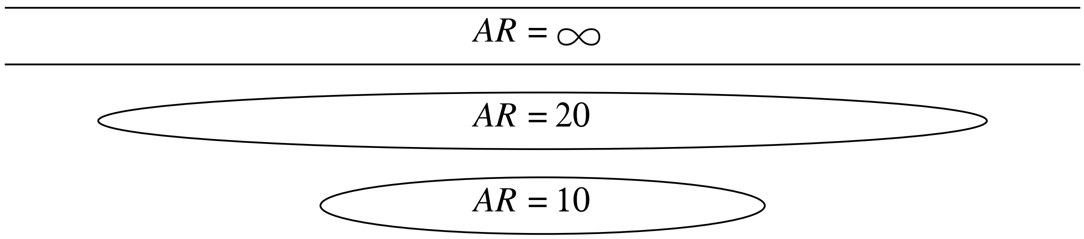
In this case $C_{Dp} = c_d$, and together with the induced drag result the total drag coefficient can then be computed as follows.

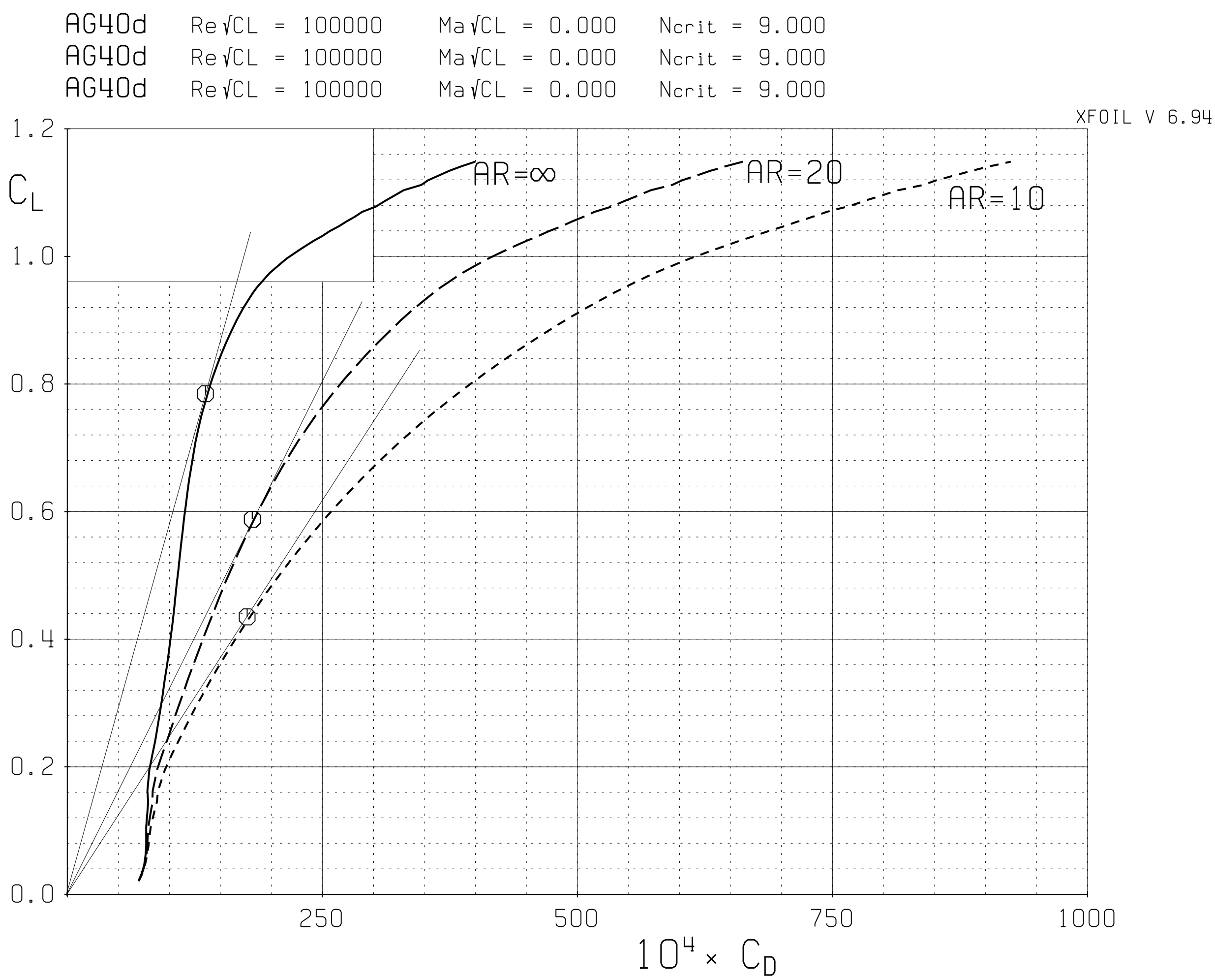
$$C_D(C_L; Re_{\mathrm{avg}}) \;=\; c_d\,(C_L; Re_{\mathrm{avg}}) \;+\; \frac{C_L^2}{\pi AR}$$

The figure shows a typical $C_D(C_L)$ polar plot for one Reynolds number and two aspect ratios, $AR = 20$ and $AR = 10$, together with the 2-D $c_d(C_L)$ curve, which can be viewed as the limiting $AR = \infty$ case. Note that this polar corresponds to the *entire wing*, rather than to just one 2-D airfoil section. Two features are immediately apparent:

1) The maximum lift/drag ratio $(C_L/C_D)_{\max}$, indicated by the slope of the tangent line, decreases considerably as AR is decreased. Since C_L/C_D is a critical aircraft performance parameter, especially for range or duration, this indicates the importance of large aspect ratio.

2) The C_L at which the maximum C_L/C_D ratio is reached decreases as AR decreases. This implies that for a given wing loading, the aircraft with the smaller aspect ratio must be flown faster to attain its best range or duration.


AR = ∞
AR = 20
AR = 10


AG40d Re√CL = 100000 Ma√CL = 0.000 Ncrit = 9.000
AG40d Re√CL = 100000 Ma√CL = 0.000 Ncrit = 9.000
AG40d Re√CL = 100000 Ma√CL = 0.000 Ncrit = 9.000
XFOIL V 6.94
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AR=∞
AR=20
AR=10
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250
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