Unified Engineering II **Spring 2004**

Problem S12 (Signals and Systems) Solution

For each signal below, find the bilateral Laplace transform (including the region of convergence) by directly evaluating the Laplace transform integral. If the signal does not have a transform, say so.

1.
$$g(t) = \sin(at)\sigma(-t)$$

To do this problem, expand the sinusoid as complex exponentials, so that

$$g(t) = \left[\frac{e^{ajt} - e^{-ajt}}{2j}\right]\sigma(-t)$$

Therefore, the LT is given by

$$G(s) = \int_{-\infty}^{0} \left[\frac{e^{ajt} - e^{-ajt}}{2j}\right] e^{-st}\,dt$$

For the LT to converge, the integrand must go to zero as t goes to $-\infty$. Therefore, the integral converges only for $\mathrm{Re}[s] < 0$. The integral is then

$$\begin{aligned}
G(s) & \int_{-\infty}^{0} \left[\frac{e^{ajt} - e^{-ajt}}{2j}\right] e^{-st}\,dt \\
&= \frac{1}{2j}\left[\frac{1}{-s+aj} e^{(aj-s)t}\Big|_{-\infty}^{0} - \frac{1}{-s-aj} e^{(-aj-s)t}\Big|_{-\infty}^{0}\right] \\
&= \frac{1}{2j}\left[\frac{1}{-s+aj} - \frac{1}{-s-aj}\right] \\
&= \frac{-a}{s^2+a^2}, \qquad \mathrm{Re}[s] < 0
\end{aligned}$$

2.
$$g(t) = te^{at}\sigma(-t)$$

The LT is given by

$$G(s) = \int_{-\infty}^{0} te^{at}e^{-st}\,dt = \int_{-\infty}^{0} te^{(a-s)t}\,dt$$

For the LT to converge, the integrand must go to zero as t goes to $-\infty$. Therefore, the integral converges only for $\mathrm{Re}[s] < a$. To find the integral, integrate by parts:

$$\begin{aligned}
G(s) &= \int_{-\infty}^{0} te^{(a-s)t}\,dt \\
&= t\frac{1}{a-s}e^{(a-s)t}\Big|_{-\infty}^{0} - \frac{1}{a-s}\int_{-\infty}^{0} e^{(a-s)t}\,dt \\
&= 0 - \frac{1}{a-s}\int_{-\infty}^{0} e^{(a-s)t}\,dt \\
&= -\frac{1}{(a-s)^2}e^{(a-s)t}\Big|_{-\infty}^{0} \\
&= -\frac{1}{(s-a)^2}, \qquad \mathrm{Re}[s] < a
\end{aligned}$$

3.

$$g(t) = \cos(\omega_0 t)\, e^{-a|t|}, \qquad \text{for all } t$$

The LT is given by

$$G(s) = \int_{-\infty}^{\infty} \cos(\omega_0 t)\, e^{-a|t|}e^{-st}\,dt$$

For the LT to converge, the integrand must go to zero as t goes to $-\infty$ and ∞. Therefore, the integral converges only for $-a < \mathrm{Re}[s] < a$. The integral is given by

$$\begin{aligned}
G(s) &= \int_{-\infty}^{\infty} \cos(\omega_0 t)\, e^{-a|t|}e^{-st}\,dt \\
&= \int_{-\infty}^{0} \cos(\omega_0 t)\, e^{at}e^{-st}\,dt \quad + \quad \int_{0}^{\infty} \cos(\omega_0 t)\, e^{-at}e^{-st}\,dt
\end{aligned}$$

Expanding the cosine term as

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

yields

$$\begin{aligned}
G(s) &= \int_{-\infty}^{0} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\, e^{at}e^{-st}\,dt \quad + \quad \int_{0}^{\infty} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\, e^{-at}e^{-st}\,dt \\
&= \int_{-\infty}^{0} \frac{e^{(j\omega_0+a-s)t} + e^{(-j\omega_0+a-s)t}}{2}\,dt \quad + \quad \int_{0}^{\infty} \frac{e^{(j\omega_0-a-s)t} + e^{(-j\omega_0-a-s)t}}{2}\,dt \\
&= \frac{1}{2}\left[\frac{1}{j\omega_0+a-s} + \frac{1}{-j\omega_0+a-s} - \frac{1}{j\omega_0-a-s} - \frac{1}{-j\omega_0-a-s}\right] \\
&= \frac{s+a}{s^2+2as+a^2+\omega_0^2} - \frac{s-a}{s^2-2as+a^2+\omega_0^2}, \qquad -a < \mathrm{Re}[s] < a
\end{aligned}$$