F4 – Lecture Notes

1. Thin Airfoil Theory Application: Analysis Example

Reading: Anderson 4.8, 4.9

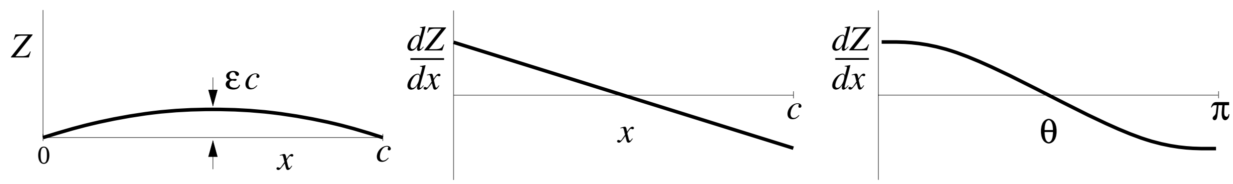
Analysis Example

Airfoil camberline definition
Consider a thin airfoil with a simple parabolic-arc camberline, with a maximum camber height εc.

$$Z(x) \;=\; 4\varepsilon\, x \left(1 - \frac{x}{c}\right)$$

The camberline slope is then a linear function in x, or a cosine function in θ.

$$\frac{dZ}{dx} \;=\; 4\varepsilon \left(1 - 2\frac{x}{c}\right) \;=\; 4\varepsilon\, \cos\theta_o$$



Fourier coefficient calculation
Substituting the above dZ/dx into the general expressions for the Fourier coefficients gives

$$A_0 \;=\; \alpha \;-\; \frac{1}{\pi}\int_0^{\pi} \frac{dZ}{dx}\, d\theta \;=\; \alpha \;-\; \frac{1}{\pi}\int_0^{\pi} 4\varepsilon\, \cos\theta\, d\theta$$

$$A_n \;=\; \frac{2}{\pi}\int_0^{\pi} \frac{dZ}{dx}\, \cos n\theta\, d\theta \;=\; \frac{2}{\pi}\int_0^{\pi} 4\varepsilon\, \cos\theta\, \cos n\theta\, d\theta$$

The integral in the A_0 expression easily evaluates to zero. The integral in the A_n expression can be evaluated by using the *orthogonality property* of the cosine functions.

$$\int_0^{\pi} \cos n\theta\, \cos m\theta\, d\theta \;=\; \begin{cases} \pi & (\text{if } n = m = 0) \\ \pi/2 & (\text{if } n = m \neq 0) \\ 0 & (\text{if } n \neq m) \end{cases}$$

For our case we have $m = 1$, and then set $n = 1, 2, 3 \ldots$ to evaluate $A_1, A_2, A_3, \ldots$. The final results are

$$\begin{aligned} A_0 &= \alpha \\ A_1 &= 4\varepsilon \\ A_2 &= 0 \\ A_3 &= 0 \\ &\vdots \end{aligned}$$

so only A_0 and A_1 are nonzero for this case.

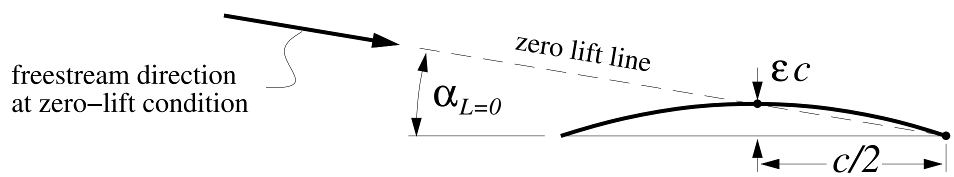
Lift and moment coefficients
The coefficients can now be computed directly using their general expressions derived previously.

$$c_\ell \;=\; \pi\,(2A_0 \,+\, A_1) \;=\; 2\pi\,(\alpha \,+\, 2\varepsilon)$$
$$c_{m,c/4} \;=\; \frac{\pi}{4}\,(A_2 \,-\, A_1) \;=\; -\pi\varepsilon$$

From the $c_\ell(\alpha)$ expression above, the zero-lift angle is seen to be

$$\alpha_{L=0} \;=\; -2\varepsilon$$

which is also the angle of the *zero lift line*. In the present case of a parabolic camber line, the zero lift line passes through the maximum-camber point and the trailing edge point.

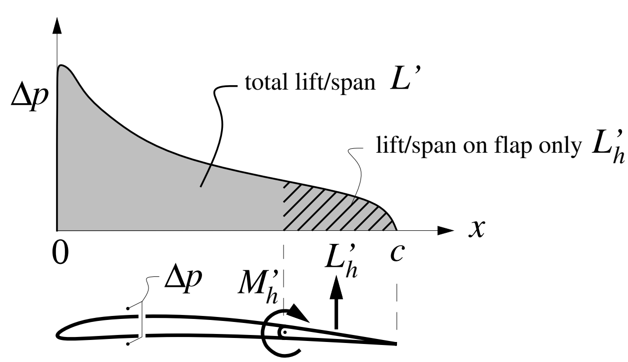


As a possible shortcut, the zero-lift angle could also have been computed directly from its explicit equation derived earlier.

$$\alpha_{L=0} \;=\; \frac{1}{\pi}\int_0^\pi \frac{dZ}{dx}\,(1-\cos\theta_o)\;d\theta_o \;=\; \frac{1}{\pi}\int_0^\pi 4\varepsilon\;\cos\theta_o\,(1-\cos\theta_o)\;d\theta_o \;=\; -2\varepsilon$$

But this integral is just the combination of the integrals for A_0 and A_1, so there is no real simplification here.

Surface loading (further details)
In many applications, obtaining just the c_ℓ and c_m of the entire airfoil is sufficient. But in some cases, we may also want to know the force and moment on only a portion of the airfoil. For example, the force and moment on a flap are of considerable interest, since the flap hinge and flap control linkage must be designed to withstand these loads. We therefore need to know how the loading $\Delta p(x)$ is distributed over the chord, and over the flap in particular.



The loading Δp is directly related to the vortex sheet strength $\gamma(x)$, and can also be given in terms of the dimensionless pressure coefficient.

$$\Delta p(x) \;=\; \rho\,V_\infty\gamma(x) \;=\; \frac{1}{2}\rho\,V_\infty^2\,\Delta C_p(x) \tag{1}$$

The general expression for the sheet strength, obtained previously, is

$$\gamma(\theta) \;=\; 2V_\infty \left(A_0 \frac{1+\cos\theta}{\sin\theta} \;+\; \sum_{n=1}^{N} A_n \sin n\theta \right)$$

Substituting the Fourier coefficients obtained for the present case gives

$$\gamma(\theta) \;=\; 2V_\infty \left(\alpha \frac{1+\cos\theta}{\sin\theta} \;+\; 4\varepsilon \sin\theta \right)$$

$$\text{or} \qquad \Delta C_p(\theta) \;=\; 2\frac{\gamma(\theta)}{V_\infty} \;=\; 4\alpha \frac{1+\cos\theta}{\sin\theta} \;+\; 16\varepsilon \sin\theta$$

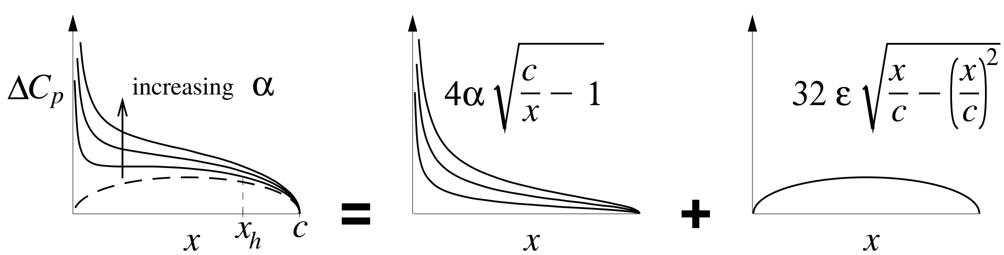
The integration of ΔC_p over the flap can be conveniently performed in the θ coordinate as usual, using the above expression. But it is also of some interest to examine this distribution in the physical x coordinate. The relevant relations between θ and x are

$$\cos\theta \;=\; 1 \,-\, 2x/c$$

$$\sin\theta \;=\; \sqrt{1-\cos^2\theta} \;=\; \sqrt{1-(1-2x/c)^2} \;=\; 2\sqrt{x/c-(x/c)^2}$$

which can be substituted into the above $\Delta C_p(\theta)$ expression to put it in terms of x.

$$\Delta C_p(x) \;=\; 4\alpha \sqrt{\frac{c}{x} - 1} \;+\; 32\varepsilon \sqrt{\frac{x}{c} - \left(\frac{x}{c}\right)^2}$$



Define x_h as the location of the flap hinge, so the flap extends from $x = x_h$, to the trailing edge at $x = c$. The corresponding θ locations are $\theta = \arccos(1-2x_h/c) \equiv \theta_h$, and $\theta = \pi$, respectively. The load/span and moment/span coefficients on the flap hinge can now be computed by integrating the pressure loading.

$$c_{\ell_h} \;\equiv\; \frac{L'_h}{\frac{1}{2}\rho V_\infty^2 c} \;=\; \frac{1}{c}\int_{x_h}^{c} \Delta C_p(x)\, dx \;=\; \frac{1}{2}\int_{\theta_h}^{\pi} \Delta C_p(\theta)\, \sin\theta\, d\theta$$

$$c_{m_h} \;\equiv\; \frac{M'_h}{\frac{1}{2}\rho V_\infty^2 c^2} \;=\; \frac{1}{c^2}\int_{x_h}^{c} \Delta C_p(x)\,(x_h - x)\, dx \;=\; \frac{1}{4}\int_{\theta_h}^{\pi} \Delta C_p(\theta)\,(\cos\theta - \cos\theta_h)\, \sin\theta\, d\theta$$

Here, integrations in θ are simpler, but still somewhat tedious, and are best left for symbolic integration methods such as Maple.