LECTURE 012

Partial Fraction Expansion (continued)

Numerator Order ≥ Denominator Order

What happens when $m \geq n$?

$m = n$:

Example

$$G(s) = \frac{2s^2 + 6s + 5}{s^2 + 3s + 2}$$

$$= \frac{2s^2 + 6s + 5}{(s+1)(s+2)}$$

$G(s)$ can be expanded as

$$G(s) = C_0 + \frac{c_1}{s+1} + \frac{c_2}{s+2}$$

What is c_0? Equations above must be true as $s \to \infty$.

$$\lim_{s \to \infty} G(s) = \frac{2}{1} = 2 \qquad (s^2 >> s,\ s^2 >> 1)$$

or, use L'Hôpital's rule. Also,

$$\lim_{s \to \infty} G(s) = c_0 + \frac{c_1}{\infty} + \frac{c_2}{\infty} = c_0$$

That is, the constant term is the ratio of the coefficient of the leading term in the numerator to the coefficient of the leading term in the denominator.

Remaing terms can be found by coverup method:

$$C_1 = \left.\frac{2s^2+6s+5}{s+2}\right|_{s=-1} = \frac{1}{1} = 1$$

$$C_2 = \left.\frac{2s^2+6s+5}{s+1}\right|_{s=-2} = \frac{1}{-1} = -1$$

$$\Rightarrow G(s) = 2 + \frac{1}{s+1} - \frac{1}{s+2}$$

$$\Rightarrow g(t) = 2\delta(t) + e^{-t}\sigma(t) - e^{-2t}\sigma(t)$$

$m > n$:

<u>Example</u> $$G(s) = \frac{2s^2+5s+5}{s+1}$$

$$= c_0 s + c_1 + \frac{c_2}{s+1}$$

Find constants by long division:

$$\begin{array}{r|l} & \quad 2s + 3 \\ \hline s+1 & 2s^2 + 5s + 5 \\ & -(2s^2 + 2s) \\ \hline & \quad 3s + 5 \\ & \quad 3s + 3 \\ \hline & \qquad 2 \end{array}$$

$\leftarrow$ remainder.

$$\Rightarrow G(s) = 2s + 3 + \frac{2}{s+1}$$

[For higher order problem, would need to do coverup method on either $G(s)$ or remainder term]

$$\Rightarrow g(t) = 2\dot{\delta}(t) + 3\delta(t) + 2e^{-t}\sigma(t)$$

"doublet," which is really a differentiator.

The Laplace Transform of a Convolution

Basic result:

$$\mathcal{L}[g(t) * u(t)] = G(s)\,U(s) = Y(s)$$

Two ways to show this:

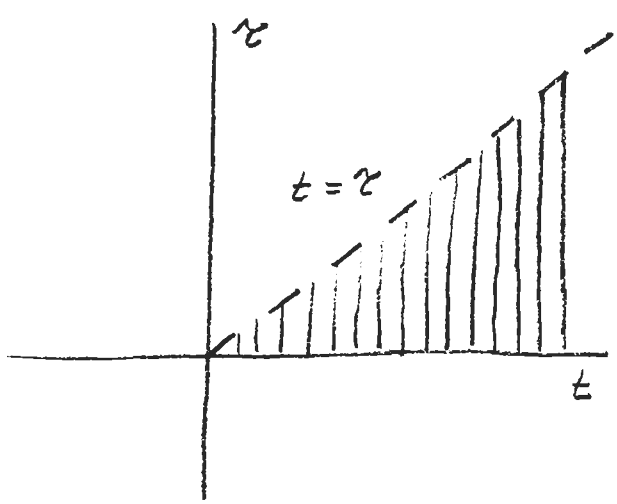
1. Direct Integration

$$Y(s) = \mathcal{L}[g(t) * u(t)]$$

$$= \int_0^\infty \left[\int_0^t g(t-\tau)\,u(\tau)\,d\tau\right] e^{-st}\,dt$$

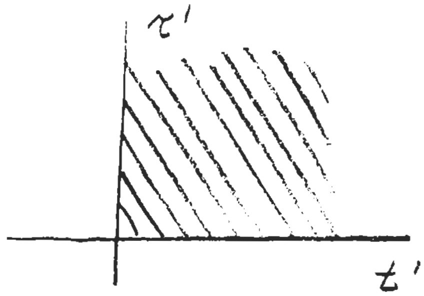
$$= \int_0^\infty \int_0^t g(t-\tau)\,u(\tau)\,e^{-st}\,d\tau\,dt$$

The region of integration is triangular:



Change variables to make integration area "square"

$$\left.\begin{aligned} t' &= t - \tau \\ \tau' &= \tau \end{aligned}\right\} \quad \begin{aligned} \tau &= \tau' \\ t &= t' + \tau' \end{aligned}$$



Change of variables in two dimensions:

$$\begin{pmatrix} t' \\ \tau' \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} t \\ \tau \end{pmatrix}$$

$$dt'\,d\tau' = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} dt\,d\tau = dt\,d\tau$$

(More generally, need the Jacobian of the transformation)

Therefore,

$$Y(s) = \int_0^\infty \int_0^\infty g(t')\,u(\tau')\,e^{-s(t'+\tau')}\,dt'\,d\tau'$$

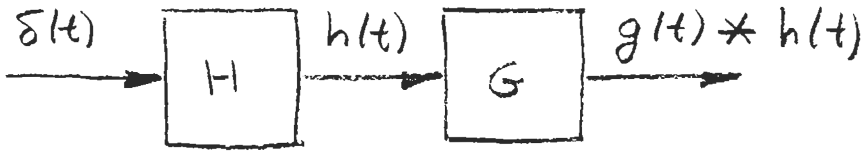
$$= \int_0^\infty \int_0^\infty g(t)\,u(\tau)\,e^{-st}\,e^{-s\tau}\,dt'\,d\tau'$$

$$= \int_0^\infty g(t) e^{-st}\, dt \int_0^\infty u(\tau) e^{-s\tau}\, d\tau$$

$$= G(s)\, U(s)$$

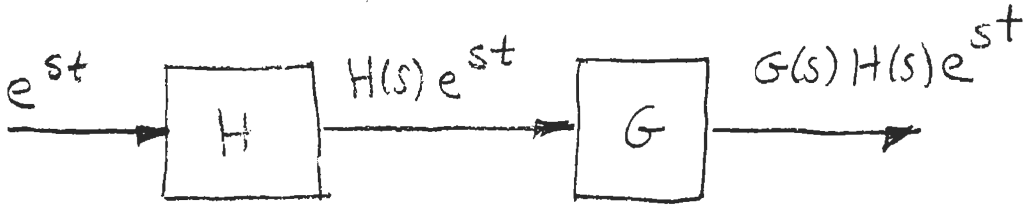
Note: the above derivation is valid so long as each integral is <u>absolutely</u> <u>convergent</u>

2. Use properties of linear systems:



So impulse response is $g(t) * h(t)$.

What is transfer function?

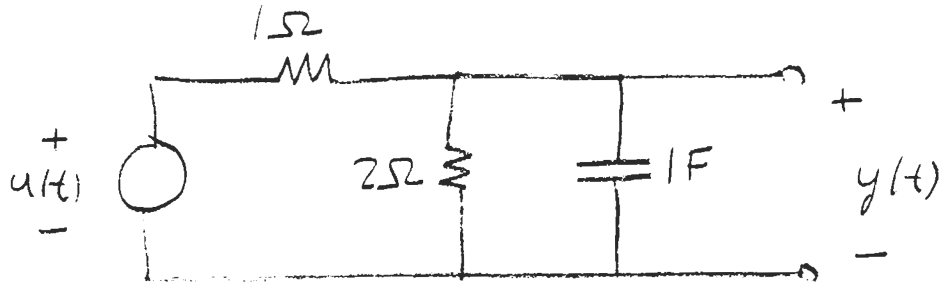


So transfer function is $G(s)\, H(s)$

$$\Rightarrow \mathcal{L}\left[g(t) * h(t) \right] = G(s)\, H(s).$$

Easy!

Example Find the response of the circuit



to the input $u(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$G(s)$ = transfer function

$$= \frac{2 \| \frac{1}{s}}{2 \| \frac{1}{s} + 1} \quad \text{(voltage divider)}$$

$$2 \| 1/s = \frac{2/s}{2 + 1/s} = \frac{2}{2s+1}$$

$$G(s) = \frac{\frac{2}{2s+1}}{\frac{2}{2s+1} + 1} = \frac{2}{2 + 2s + 1}$$

$$= \frac{2}{2s+3} = \frac{1}{s+1.5}$$

$$U(s) = \mathcal{L}\left[u(t)\right] = \frac{1}{s+1}$$

$$Y(s) = G(s)\,U(s)$$

$$= \frac{1}{s+1.5}\,\frac{1}{s+1}$$

$$= \frac{2}{s+1} - \frac{2}{s+1.5}$$

$$\Rightarrow \quad y(t) = \left[2e^{-t} - 2e^{-1.5t}\right]\sigma(t)$$