F19 – Lecture Notes

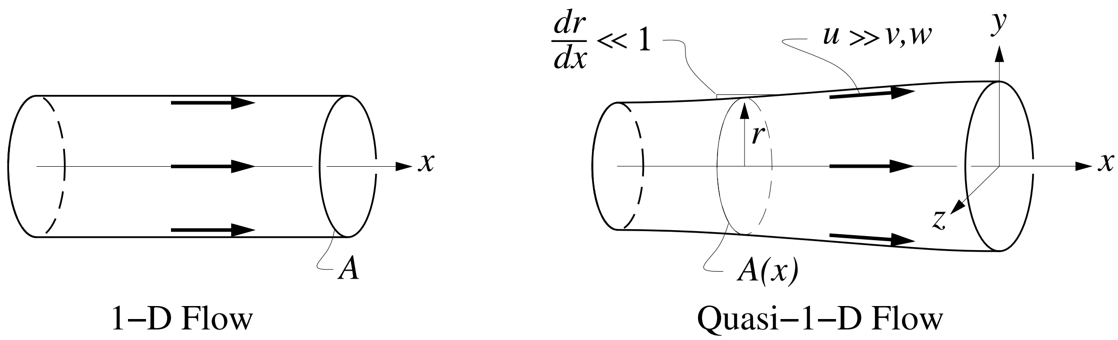
1. Compressible Channel Flow

Reading: Anderson 10.1, 10.2

Compressible Channel Flow

Quasi-1-D Flow

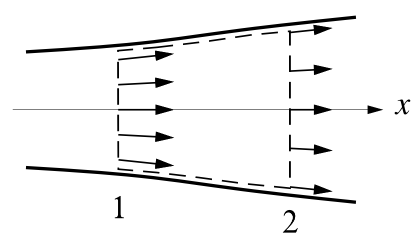
A *quasi-one-dimensional flow* is one in which all variables vary primarily along one direction, say x. A flow in a duct with slowly-varying area $A(x)$ is the case of interest here. In practice this means that the slope of the duct walls is small. Also, the x-velocity component u dominates the y and z-components v and w.



Governing equations

Application of the integral mass continuity equation to a segment of the duct bounded by any two x locations gives

$$\begin{aligned} \oiint \rho \vec{V} \cdot \hat{n} \, dA &= 0 \\ -\rho_1 u_1 \iint_1 dA \; + \; \rho_2 u_2 \iint_2 dA &= 0 \\ -\rho_1 u_1 A_1 \; + \; \rho_2 u_2 A_2 &= 0 \end{aligned}$$



The quasi-1-D approximation is invoked in the second line, with u and ρ assumed constant on each cross-sectional area, so they can be taken out of the area integral.

Since stations 1 or 2 can be placed at any arbitrary location x, we can define the duct *mass flow* which is constant all along the duct, and relates the density, velocity, and area.

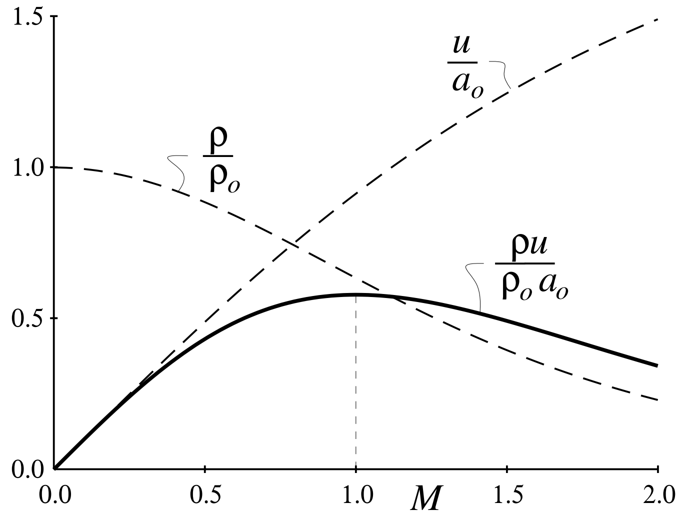
$$\rho(x) \, u(x) \, A(x) \; \equiv \; \dot{m} \; = \; \text{constant} \tag{1}$$

If we assume that the flow in the duct is isentropic, at least piecewise-isentropic between shocks, the stagnation density ρ_o and stagnation speed of sound a_o are both constant. This allows the normalized ρ and u to be given in terms of the Mach number alone.

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{1}{\gamma-1}} \tag{2}$$

$$\frac{u}{a_o} = \frac{M a}{a_o} = M \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{1}{2}} \tag{3}$$

$$\frac{\rho u}{\rho_o a_o} = M \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \tag{4}$$



The figure shows these variables, along with the normalized *mass flux*, or ρu product, all plotted versus Mach number.

The significance of ρu is that it represents the inverse of the duct area, or

$$A \sim \frac{1}{\rho u}$$

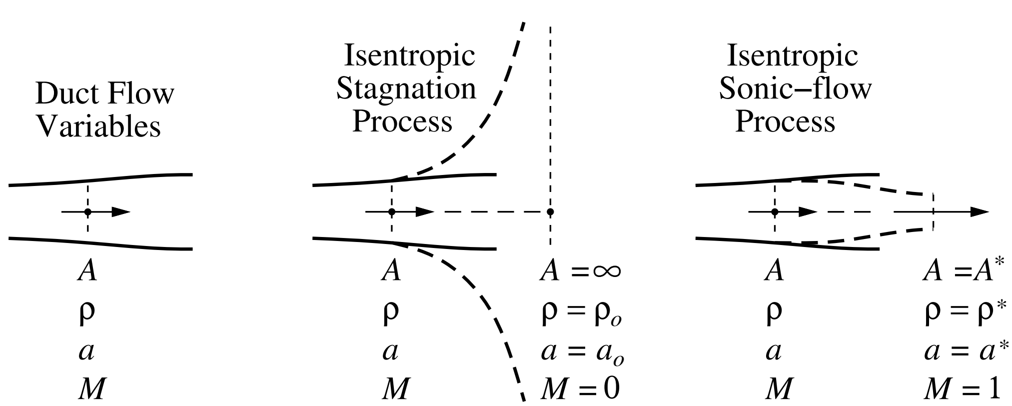
It is evident that the maximum possible mass flux occurs at a location where locally $M = 1$. This can be proven by computing

$$\frac{d}{dM}\left(\frac{\rho u}{\rho_o a_o}\right) = \left(1 - M^2\right)\left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{\gamma-3}{2(\gamma-1)}}$$

which is clearly zero at $M = 1$. Therefore, the duct must have a local minimum, or *throat*, wherever $M = 1$.

Sonic conditions

In the development above, the stagnation conditions ρ_o and a_o were used to normalize the various quantities. For compressible duct flows, it is very convenient to also define *sonic conditions* which can serve as alternative normalizing quantities. These are defined by a hypothetical process where the flow is sent through a duct of progressively reduced area until $M = 1$ is reached, shown in the figure along with the familiar stagnation process.



The resulting quantities at the hypothetical sonic throat are denoted by a $()^*$ superscript. The advantage of the sonic-flow process is that it produces a well-defined *sonic throat area*

A^*, while for the stagnation process A tends to infinity, and cannot be used for normalization. The ratios between the stagnation and sonic conditions are readily obtained from the usual isentropic relations, with $M = 1$ plugged in. Numerical values are also given for $\gamma = 1.4$.

$$\frac{\rho^*}{\rho_o} = \left(1+\frac{\gamma-1}{2}\right)^{-\frac{1}{\gamma-1}} = 0.6339$$
$$\frac{a^*}{a_o} = \left(1+\frac{\gamma-1}{2}\right)^{-\frac{1}{2}} = 0.9129$$
$$\frac{p^*}{p_o} = \left(1+\frac{\gamma-1}{2}\right)^{-\frac{\gamma}{\gamma-1}} = 0.5283$$

The sonic flow area A^* can be obtained from the constant mass flow equation (1). For the sonic-flow process we have

$$\dot{m} = \rho u A = \rho^* u^* A^*$$

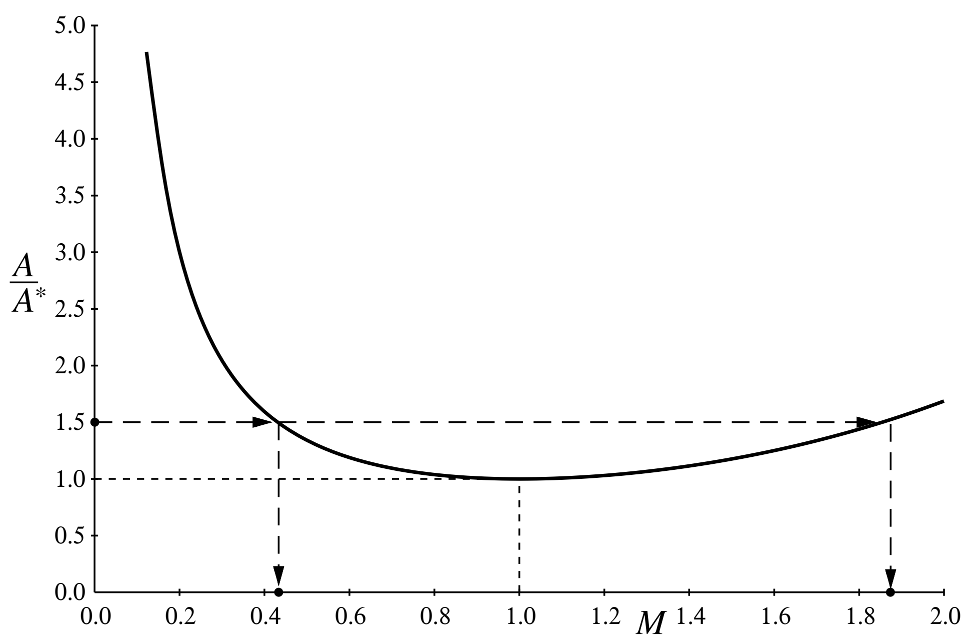
and we also note that $u^* = a^*$ since $M = 1$ at the sonic throat. Therefore,

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{a_o}\frac{a_o}{u}$$

Using the previously-defined expressions produces

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \qquad (5)$$

This is the *area-Mach relation*, which is plotted in the figure below for $\gamma = 1.4$, and is also available in tabulated form. It uniquely relates the local Mach number to the area ratio A/A^*, and can be used to "solve" compressible duct flow problems. If the duct geometry $A(x)$ is given, and A^* is defined from the known duct mass flow and stagnation quantities, then $M(x)$ can be determined using the graphical technique shown in the figure, or using the equivalent numerical table.



Once $M(x)$ is determined, any remaining quantity of interest, such as $\rho(x)$, $u(x)$, $p(x)$, etc., can be computed from the isentropic or adiabatic relations such as (2) and (3).

Note that for any given area $A(x)$, two solutions are possible for the given mass flow: a subsonic solution with $M < 1$, and a supersonic solution with $M > 1$. Which solution corresponds to the actual flow depends on whether the flow upstream of that x location is subsonic or supersonic.

There is also the possibility of shock waves appearing in the duct. This introduces additional complications which will be considered later.