F8 – Lecture Notes

1. Momentum Flow
2. Momentum Conservation

Reading: Anderson 2.5

Momentum Flow

Before we can apply the principle of momentum conservation to a fixed permeable control volume, we must first examine the effect of flow through its surface. When material flows through the surface, it carries not only mass, but momentum as well. The *momentum flow* can be described as

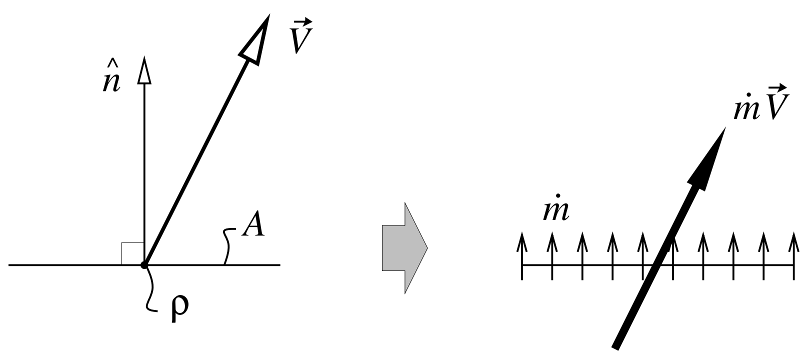
$$\overrightarrow{\text{momentum}} \text{ flow} = \text{(mass flow)} \times (\overrightarrow{\text{momentum}} \text{ /mass})$$

where the mass flow was defined earlier, and the momentum/mass is simply the velocity vector $\vec{V}$. Therefore

$$\overrightarrow{\text{momentum}} \text{ flow} = \dot{m}\,\vec{V} \;=\; \rho\left(\vec{V}\cdot\hat{n}\right)A\,\vec{V} \;=\; \rho\,V_n A\,\vec{V}$$

where $V_n = \vec{V}\cdot\hat{n}$ as before. Note that while mass flow is a scalar, the momentum flow is a vector, and points in the same direction as $\vec{V}$. The momentum flux vector is defined simply as the momentum flow per area.

$$\overrightarrow{\text{momentum}} \text{ flux} = \rho\,V_n\,\vec{V}$$



Momentum Conservation

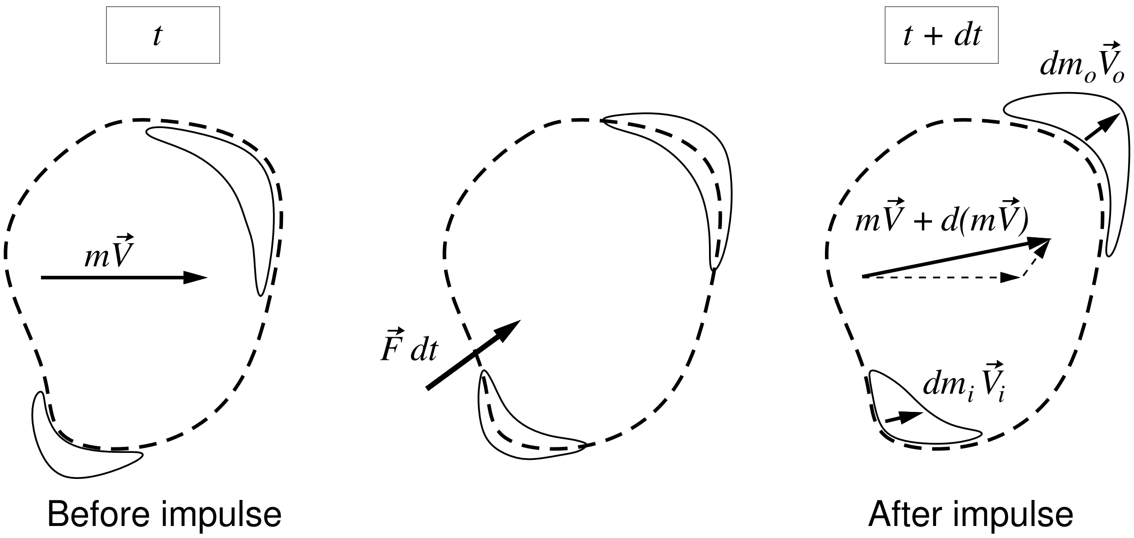
The most general statement of Newton's second law is

$$\vec{F} \;=\; \frac{d}{dt}\left(m_a\vec{V}\right)$$

which states that a force applied to affected mass m_a results in a rate of increase of momentum of that mass. For a fixed control volume situation, however, the affected mass m_a in the equation above involves not only to the mass inside the volume, but to mass which is in the process of flowing in or out of the volume.

The figure shows what happens during an infinitesimal time interval dt during which a force $\vec{F}$ is applied to the fluid in the fixed control volume. The resulting impulse $\vec{F}\,dt$ involves three pieces of momentum change:

$$
\begin{aligned}
d\left(m\vec{V}\right) &= \text{change in momentum inside the volume} \\
dm_o\,\vec{V}_o &= \text{momentum of mass } dm_o \text{ which was inside volume, and moved out} \\
dm_i\,\vec{V}_i &= \text{momentum of mass } dm_i \text{ which was outside volume, and moved in}
\end{aligned}
$$



The impulse is applied to the material which is inside the volume at initial time t. Proper momentum accounting then adds the dm_o part, and removes the dm_i part.

$$\vec{F}\,dt \;=\; d\left(m\vec{V}\right) \;+\; dm_o\,\vec{V}_o \;-\; dm_i\,\vec{V}_i$$

The equivalent equation in terms of rates of change is

$$\vec{F} \;=\; \frac{d}{dt}\left(m\vec{V}\right) \;+\; \dot{m}\,\vec{V} \tag{1}$$

where the $\dot{m}$ term combines the outflow and inflow terms, with the understanding that $\dot{m} > 0$ for outflow, and $\dot{m} < 0$ for inflow.

Applied forces

The force $\vec{F}$ can be of two types.

Body forces. These act on fluid inside the volume. The most common example is the gravity force, along the gravitational acceleration vector $\vec{g}$.

$$\vec{F}_{\text{gravity}} \;=\; \iiint \rho\,\vec{g}\,d\mathcal{V}$$

Surface forces. These act on the surface of the volume, and can be separated into pressure and viscous forces.

$$\vec{F}_{\text{pressure}} \;=\; \oiint -p\,\hat{n}\,dA$$

The viscous force is omplicated to write out, and for simplicity will simply be called $\vec{F}_{\text{viscous}}$.

The remaining terms in equation (1) are given as volume and surface integrals.

Integral Momentum Equation

Expressing all terms in the momentum statement (1) as either surface or volume integrals gives the *Integral Momentum Equation*.

$$\frac{d}{dt} \iiint \rho \vec{V} \, d\mathcal{V} \; + \; \oiint \rho \left(\vec{V} \cdot \hat{n} \right) \vec{V} \, dA \; + \; \oiint p \, \hat{n} \, dA \; = \; \iiint \rho \, \vec{g} \, d\mathcal{V} \; + \; \vec{F}_{\text{viscous}} \qquad (2)$$

Along with the Integral Mass Equation, this equation can be applied to solve many problems involving finite control volumes.

Differential Momentum Equation

The pressure surface integral in equation (2) can be converted to a volume integral using the Gradient Theorem.

$$\oiint p \, \hat{n} \, dA \; = \; \iiint \nabla p \, d\mathcal{V}$$

The mass-flow surface integral is also similarly converted using Gauss's Theorem. This integral is a vector quantity, and for clarity the conversion is best done on each component separately. After substituting $\vec{V} = u \, \hat{\imath} + v \, \hat{\jmath} + w \, \hat{k}$, we have

$$\begin{aligned} \oiint \rho \left(\vec{V} \cdot \hat{n} \right) \left(u \, \hat{\imath} \, + \, v \, \hat{\jmath} \, + \, w \, \hat{k} \right) dA \; &= \; \hat{\imath} \iiint \nabla \cdot \left(\rho \vec{V} u \right) \, d\mathcal{V} \\ &+ \; \hat{\jmath} \iiint \nabla \cdot \left(\rho \vec{V} v \right) \, d\mathcal{V} \\ &+ \; \hat{k} \iiint \nabla \cdot \left(\rho \vec{V} w \right) \, d\mathcal{V} \end{aligned}$$

The x-component of the integral momentum equation (2) can not be written strictly in terms of volume integrals.

$$\iiint \left[\frac{\partial (\rho u)}{\partial t} \; + \; \nabla \cdot \left(\rho u \vec{V} \right) \; + \; \frac{\partial p}{\partial x} \; - \; \rho g_x \; - \; (F_x)_{\text{viscous}} \right] d\mathcal{V} \; = \; 0 \qquad (3)$$

This relation must hold for *any* control volume whatsoever. If we place an infinitesimal control volume at every point in the flow and apply the above equation, we can see that the whole quantity in the brackets must be zero at every point. This results in the *x-Momentum Equation*

$$\frac{\partial (\rho u)}{\partial t} \; + \; \nabla \cdot \left(\rho u \vec{V} \right) \; = \; -\frac{\partial p}{\partial x} \; + \; \rho g_x \; + \; (F_x)_{\text{viscous}} \qquad (4)$$

and the y- and z-Momentum Equations follow by the same process.

$$\frac{\partial (\rho v)}{\partial t} \; + \; \nabla \cdot \left(\rho v \vec{V} \right) \; = \; -\frac{\partial p}{\partial y} \; + \; \rho g_y \; + \; (F_y)_{\text{viscous}} \qquad (5)$$

$$\frac{\partial (\rho w)}{\partial t} \; + \; \nabla \cdot \left(\rho w \vec{V} \right) \; = \; -\frac{\partial p}{\partial z} \; + \; \rho g_z \; + \; (F_z)_{\text{viscous}} \qquad (6)$$

These three equations are the embodiment of the Newton's second law of motion, applied at every point in the flowfield. The steady flow version has the $\partial / \partial t$ terms omitted.