F13 – Lecture Notes

1. Bernoulli Equation
2. Venturi Flow

Reading: Anderson 3.2, 3.3

Bernoulli Equation

Derivation – 1-D case

The 1-D momentum equation, which is Newton's Second Law applied to fluid flow, is written as follows.

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \rho g_x + (F_x)_{\text{viscous}}$$

We now make the following assumptions about the flow.

- Steady flow: $\partial/\partial t = 0$
- Negligible gravity: $\rho g_x \simeq 0$
- Negligible viscous forces: $(F_x)_{\text{viscous}} \simeq 0$
- Low-speed flow: ρ is constant

These reduce the momentum equation to the following simpler form, which can be immediately integrated.

$$\begin{aligned}
\rho u \frac{du}{dx} + \frac{dp}{dx} &= 0 \\
\frac{1}{2}\rho \frac{d(u^2)}{dx} + \frac{dp}{dx} &= 0 \\
\frac{1}{2}\rho u^2 + p &= \text{constant} \equiv p_o
\end{aligned}$$

The final result is the one-dimensional *Bernoulli Equation*, which uniquely relates velocity and pressure if the simplifying assumptions listed above are valid. The constant of integration p_o is called the *stagnation pressure*, or equivalently the *total pressure*, and is typically set by known upstream conditions.

Derivation – 2-D case

The 2-D momentum equations are

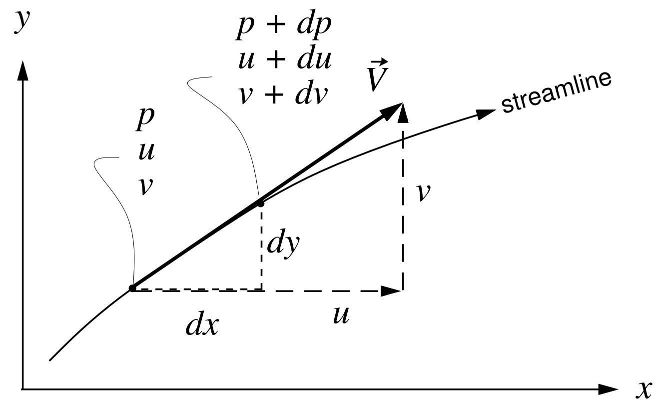
$$\begin{aligned}
\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \rho g_x + (F_x)_{\text{viscous}} \\
\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \rho g_y + (F_y)_{\text{viscous}}
\end{aligned}$$

Making the same assumptions as before, these simplify to the following.

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = 0 \tag{1}$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = 0 \tag{2}$$

Before these can be integrated, we must first restrict ourselves only to flowfield variations along a streamline. Consider an incremental distance ds along the streamline, with projections dx and dy in the two axis directions. The speed V likewise has projections u and v.



Along the streamline, we have

$$\frac{dy}{dx} = \frac{v}{u}$$

or

$$u\,dy = v\,dx \tag{3}$$

We multiply the x-momentum equation (1) by dx, use relation (3) to replace $v\,dx$ by $u\,dy$, and combine the u-derivative terms into a du differential.

$$\begin{aligned}
\rho u \frac{\partial u}{\partial x}\,dx + \rho v \frac{\partial u}{\partial y}\,dx + \frac{\partial p}{\partial x}\,dx &= 0 \\
\rho u \left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \right) + \frac{\partial p}{\partial x}\,dx &= 0 \\
\rho u\,du + \frac{\partial p}{\partial x}\,dx &= 0 \\
\frac{1}{2}\rho\,d\!\left(u^2\right) + \frac{\partial p}{\partial x}\,dx &= 0 \qquad (4)
\end{aligned}$$

We multiply the y-momentum equation (2) by dy, and performing a similar manipulation, we get

$$\begin{aligned}
\rho u \frac{\partial v}{\partial x}\,dy + \rho v \frac{\partial v}{\partial y}\,dy + \frac{\partial p}{\partial y}\,dy &= 0 \\
\rho v \left(\frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy \right) + \frac{\partial p}{\partial y}\,dy &= 0 \\
\rho v\,dv + \frac{\partial p}{\partial y}\,dy &= 0 \\
\frac{1}{2}\rho\,d\left(v^2\right) + \frac{\partial p}{\partial y}\,dy &= 0 \qquad (5)
\end{aligned}$$

Finally, we add equations (4) and (5), giving

$$\begin{aligned}
\frac{1}{2}\rho\,d\!\left(u^2 + v^2\right) + \frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy &= 0 \\
\frac{1}{2}\rho\,d\!\left(u^2 + v^2\right) + dp &= 0
\end{aligned}$$

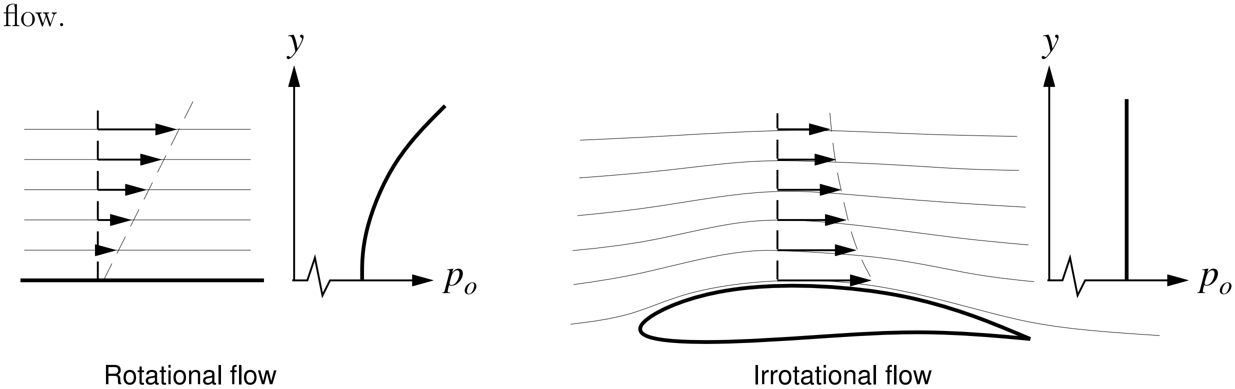
which integrates into the general Bernoulli equation

$$\frac{1}{2}\rho V^2 \;+\; p \;\;=\;\; \text{constant} \;\equiv\; p_o \qquad\qquad \text{(along a streamline)} \qquad (6)$$

where $V^2 = u^2 + v^2$ is the square of the speed. For the 3-D case the final result is exactly the same as equation (6), but now the w velocity component is nonzero, and hence $V^2 = u^2 + v^2 + w^2$.

Irrotational Flow

Because of the assumptions used in the derivations above, in particular the streamline relation (3), the Bernoulli Equation (6) relates p and V only along any given streamline. Different streamlines will in general have different p_o constants, so p and V cannot be directly related between streamlines. For example, the simple shear flow on the left of the figure has parallel flow with a linear $u(y)$, and a uniform pressure p. Its p_o distribution is therefore parabolic as shown. Hence, there is no unique correspondence between velocity and pressure in such a flow.



However, if the flow is irrotational, i.e. if $\vec{V} = \nabla\phi$ and $V^2 = |\nabla\phi|^2$, then p_o takes on the same value for all streamlines, and the Bernoulli Equation (6) becomes usable to relate p and V in the entire irrotational flowfield. Fortunately, a flowfield is irrotational if the upstream flow is irrotational (e.g. uniform), which is a very common occurance in aerodynamics. The right side of the figure shows an example. Here, if the velocity is known at any point, the pressure is known as well.

Use of Bernoulli Equation – Solving potential flows

Having the Bernoulli Equantion (6) in hand allows us to devise a relatively simple two-step solution strategy for potential flows.

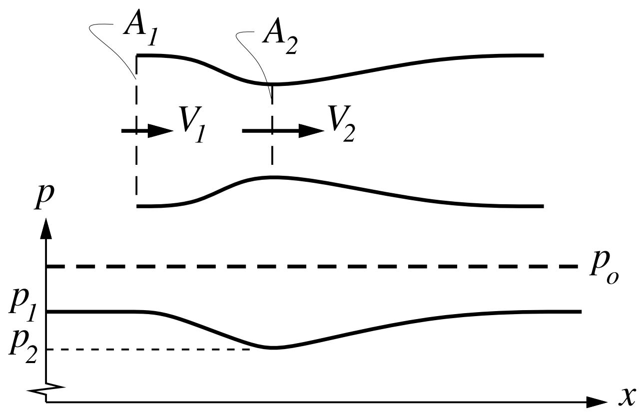
1. Determine the potential field $\phi(x, y, z)$ and resulting velocity field $\vec{V} = \nabla\phi$ using the governing equations.

2. Once the velocity field is known, insert it into the Bernoulli Equation to compute the pressure field $p(x, y, z)$.

This two-step process is simple enough to permit very economical aerodynamic solution methods which give a great deal of physical insight into aerodynamic behavior. The alternative approaches which do not rely on Bernoulli Equation must solve for $\vec{V}(x, y, z)$ and $p(x, y, z)$ simultaneously, which is a tremendously more difficult problem which can be approached only through brute force numerical computation.

Venturi Flow

One common application of the Bernoulli Equation is in a *venturi*, which is a flow tube with a minimum cross-sectional area somewhere in the middle.



Assuming incompressible flow, with ρ constant, the mass conservation equation gives

$$A_1 V_1 \;=\; A_2 V_2 \tag{7}$$

This relates V_1 and V_2 in terms of the geometric cross-sectional areas.

$$V_2 \;=\; V_1 \frac{A_1}{A_2}$$

Knowing the velocity relationship, the Bernoulli Equation then gives the pressure relationship.

$$p_1 + \frac{1}{2}\rho V_1^2 \;=\; p_o \;=\; p_2 + \frac{1}{2}\rho V_2^2 \tag{8}$$

Equations (7) and (8) together can be used to determine the inlet velocity V_1, knowing only the pressure difference $p_1 - p_2$ and the geometric areas. By direct substution we have

$$V_1 \;=\; \sqrt{\frac{2(p_1 - p_2)}{\rho\left[(A_1/A_2)^2 - 1\right]}}$$

A venturi can therefore by used as an *airspeed indicator*, if some means of measuring the pressure difference $p_1 - p_2$ is provided.