

Unified Systems Problem 3

WATER BOTTLE ROCKET PERFORMANCE ANALYSIS

Learning Objectives:

After completing this systems problem you will have:

- Applied material from 8.01, Unified thermodynamics and Unified fluid mechanics to develop a model for a single stage water rocket.
- Explored how external aerodynamics, structural weight, propellant mass fraction, internal fluid mechanics and thermodynamics jointly determine the dynamic behavior of a single stage water rocket.
- Demonstrated an ability to describe conceptually how the performance of the water rocket changes as a function of important design parameters.
- Demonstrated an ability to integrate a system of ordinary differential equations using a spreadsheet.
- Developed a preliminary design for a water rocket that you and a partner may build and test.

Overview

In this systems problem, you will estimate water rocket performance and analyze the impact of two design parameters on the maximum altitude achieved by the rocket.

For the analysis, you will begin with a baseline rocket system that uses a standard 2-litre soda bottle for the rocket structure and fuel storage. The bottle will be partially filled with water and mounted on a rocket launch mechanism as shown in Figure 1. The air in the bottle will then be pressurized, and the launch will be executed.

The launching mechanism uses a 7-inch-long rod to provide stability and to ensure that the rocket proceeds in a straight line. The bottle is inverted onto the rod, and the launcher is staked to the ground. A metal launch restraining pin is inserted over the lip of the bottle-neck to keep the bottle on the launcher until it is ready for blastoff. Air is then pumped through the rod to pressurize the bottle (slowly – assume an isothermal process); a small rubber o-ring on the launch rod provides an airtight seal at the mouth of the bottle.

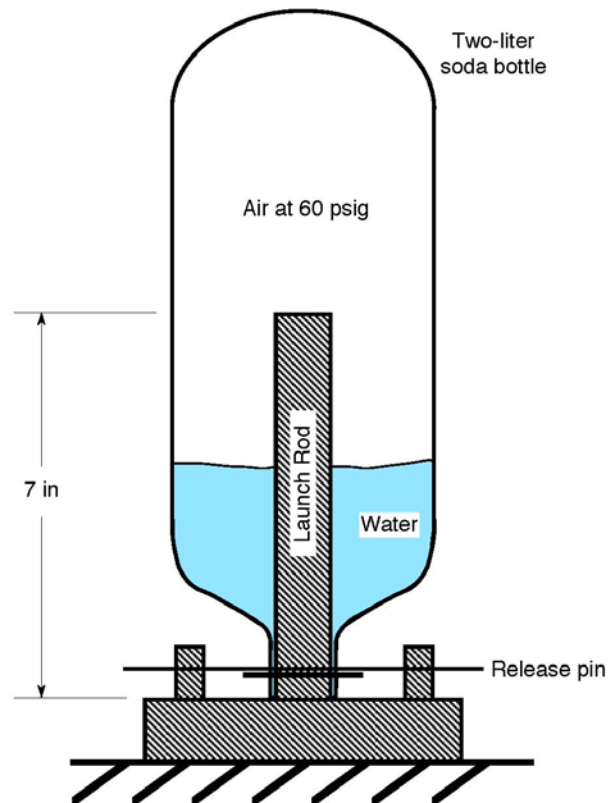


Figure 1.
Water Bottle Rocket and Launching Mechanism

The rocket launch consists of three stages. The first stage covers the period between the start of the launch until the rocket nozzle (bottle mouth) just reaches the top end of the launch rod. The second stage involves the compressed air forcing the water out of the bottle at high speed as the bottle rises into the sky. Assume that any compressed air that remains after the water is expelled produces negligible additional thrust. The third stage is a ballistic stage, in which the rocket continues upward under only the influence of gravity and drag, reaching some maximum altitude before falling back to the ground.

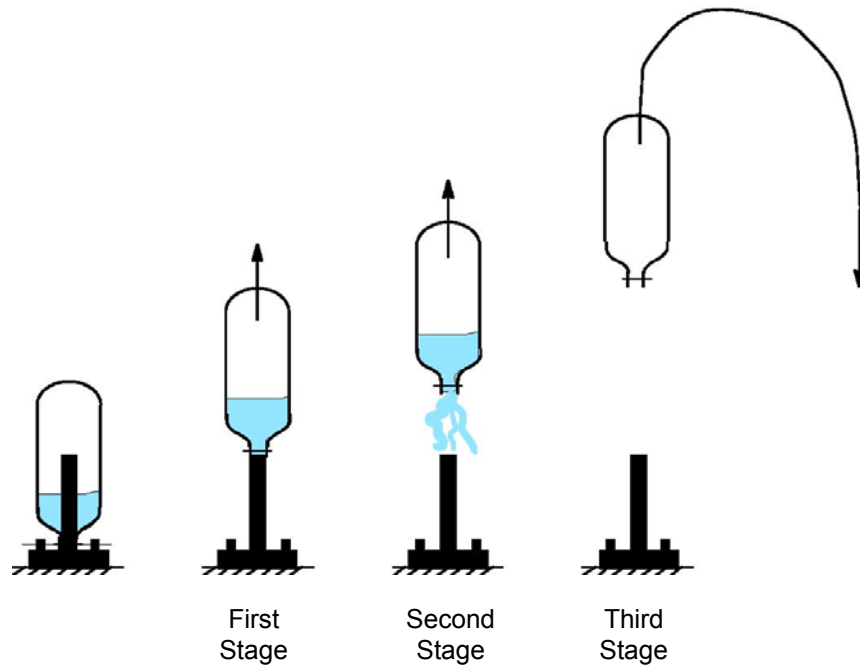


Figure 2.
Rocket Launch Stages

Table 1. Relevant parameters

Rocket		
Volume of bottle	2.3×10^{-3}	m^3
Outer diameter	1.118×10^{-1}	m
Nozzle diameter	2.159×10^{-2}	m
Empty mass	4.7×10^{-2}	kg
Coefficient of drag (estimated)	0.5	
Launcher		
Launch rod length	1.778×10^{-1}	m
Launch rod area	3.661×10^{-4}	m^2
Environmental Conditions		
Air pressure	1.013×10^5	Pa
Air temperature	288	K
Air density	1.225	kg/m^3
Water density	1×10^3	kg/m^3
Ideal gas constant	287	J/kg-K
Specific heat at constant volume	716.5	J/kg-K
Charging Conditions		
Bottle pressure	515×10^3	N/m^2
Gas temperature	288	K

1. Thermodynamic Analysis for Stage 1

For the first stage of the launch you are to assume the air in the bottle expands quasi-statically and adiabatically until the top of the launch rod is just at the mouth of the bottle. **This process will set the initial conditions for the second stage of the launch.**

In the equations below the initial conditions are labeled with the subscript “charge”. The subscript “start” is used for conditions at the start of stage 2.

Given p_{charge} and T_{charge} , with

$$V_{\text{charge}} = V_{\text{bottle}} - V_{\text{launch_rod}} - V_{\text{water_initial}}$$

$$V_{\text{start}} = V_{\text{bottle}} - V_{\text{water_initial}}$$

Calculate the work done during the adiabatic expansion from V_{charge} to V_{start} and relate this to the change in kinetic and potential energy.

This will allow you to determine the initial velocity for the second stage of the launch. (The initial height is equal to the launch rod length.)

In doing this, you have neglected any friction between the launch rod and the bottle and also any drag forces.

2. Second and third stages

For the second and third stages, you are to build upon the finite difference calculation you performed for System Problem 2.

Table 2. Nomenclature

Description	Symbol	Units
height	h	m
Rocket velocity	V	m/s
Thrust	T	N/m ²
Acceleration of gravity	g	m/s ²
Density of atmospheric air	ρ_{air}	kg/m ³
Drag coefficient	C_D	
Cross sectional area of bottle	A_{bottle}	m ²
Mass of rocket	m	kg
Density of water	ρ_{water}	kg/m ³
Exit velocity of water	u_e	m/s
Throat area of bottle	A_{throat}	m ²
Pressure in bottle	p	N/m ²
Atmospheric pressure	p_{atm}	N/m ²
Volume	Vol	m ³
Ratio of specific heats	γ	

Whereas in System Problem 2 there were 2 state variables, there are now three state variables: the height of the rocket (h), the velocity of the rocket (V) and the mass of the rocket (m).

As we will show in class, the equations governing the behavior of the rocket at time $t = t_i$ are:

$$\dot{h}_i = V_i$$

$$\dot{V}_i = \frac{T_i}{m_i} - g - \frac{1}{2} \rho_{\text{air}} V_i |V_i| \frac{C_D A_{\text{bottle}}}{m_i}$$

$$\dot{m}_i = -\rho_{\text{water}} u_e A_{\text{throat}}$$

where we have introduced the thrust, T , and the velocity of the water leaving the throat of the bottle, u_e .

Use the momentum equation, Bernoulli's equation and a quasi-static, adiabatic

expansion process for the air in the bottle to calculate the thrust and the velocity of the water leaving the bottle. Neglect the change in pressure through the water due to the weight of the water itself (the hydrostatic pressure).

You can advance the solution using the Forward Euler finite difference method you used for Systems Problem 2.

EXCEL NOTE 1: If you are using Excel you will find the function “IF(logical test, value if true, value if false)” to be useful for checking when all the water is expelled (and then setting $u_e = 0$ thereafter). This will allow you to seamlessly integrate stages 2 and 3.

EXCEL NOTE 2: You may also find the function “MAX(range of cells)” useful for returning the maximum height the rocket reaches from the column where you are integrating the height.

EXCEL NOTE 3: Use defensive and modular programming: If a spreadsheet cell contains a long and complicated formula, it is best to break it up into several smaller terms, evaluated separately in additional columns. Then the net formula is simply a combination of the separate terms. If the program does not work right, it is much easier to check the small individual terms one by one, than when they are all combined.

3. Analysis and reporting

- a. Write the equations that you used to define the initial conditions for Stage 2. Write the equations that you used to calculate the exit velocity of the water as a function of design parameters (Table 1), operating parameters and state variables. Write the finite difference equations that you integrated with the spreadsheet. [20 pts]
- b. For the baseline conditions that are given in the Table 1, determine the fraction of bottle volume that should be filled with water to maximize the height of the rocket. Call this a propellant-optimized design. In a concise, clearly written paragraph, explain why the performance becomes worse when there is either more or less water than this amount. Demonstrate your understanding of the fundamental concepts that determine the rocket performance. [20 pts]
- c. Change the drag coefficient by $\pm 10\%$ and $\pm 25\%$. Change the empty weight by $\pm 10\%$ and $\pm 25\%$. For each of these designs, use the spreadsheet to determine the height that a propellant-optimized rocket will go. Display these estimates in a simple table that is properly labeled. [20 pts]
- d. Provide a preliminary sketch of a water rocket that you may design and test for Systems Problem 3. Be sure to label your sketch. You may not modify the bottle(s) by cutting, but you may use multiple bottles, different sizes of bottles, fairings to reduce drag, etc. Include a list of key design features and provide supporting arguments for each of your design choices. Also include an estimate of the maximum height that the rocket will reach, assuming that you are limited to the same charging pressure (515×10^3 Pa). [40 pts]

Grading Rubric for Systems Problem 3

Student name: _____

	0 not at all	1 poorly done	2 adequate	3 well done	4 excellent
Does the student correctly derive the equations and initial conditions that are to be used to estimate the performance of the rocket? [part (a) = 20 pts]					
Does the student provide a correct estimate of the rocket performance reflecting satisfactory implementation of the model and finite difference method in the spreadsheet? [part (b) = 5 points]					
Does the student clearly explain the relationship between the volume of water and the rocket performance in a way that reflects an understanding of the fundamental concepts and their relationship to one another? [part (b) = 15 pts]					
Does the student change the drag coefficient and empty weight as instructed and display these data in a table? [part (c) = 5 pts]					
Are the estimates in the table correct reflecting satisfactory implementation of the model and finite difference method in the spreadsheet? [part (c) = 15 pts]					
Does the student provide a labeled sketch of a water rocket? [part (d) = 15 pts]					
Does the student estimate the maximum height? [part (d) = 5 pts]					
Does the student provide a short rationale for each design choice that is well founded based on an understanding of the fundamental concepts? [part (d) = 20 pts]					

Total points: _____