1. $$G(s) = \int_0^\infty t\, e^{-at}\, e^{-st}\, dt$$
$$= \int_0^\infty t\, e^{-(s+a)t}\, dt$$

Integrate by parts:

$$u = t \qquad\qquad du = dt$$
$$dv = e^{-(s+a)t}\, dt \quad \Longrightarrow \quad V = \frac{-1}{s+a} e^{-(s+a)t}$$

Therefore,

$$G(s) = uv - \int v\, du$$
$$= \frac{-t}{s+a} e^{-(s+a)t} \Big|_0^\infty + \frac{1}{s+a} \int_0^\infty e^{-(s+a)t}\, dt$$
$$= 0 + \frac{1}{(s+a)^2} \quad \text{if} \quad Re[s] > -a$$

$$\boxed{G(s) = \frac{1}{(s+a)^2}, \quad Re[s] > -a}$$

2. $$G(s) = \int_0^\infty \underbrace{t^2}_{u} \underbrace{e^{-at}\, e^{-st}\, dt}_{dv}$$

$$du = 2t\, dt, \quad v = \frac{-1}{s+a} e^{-(s+a)t}$$

Integrating by parts,

$$G(s) = \underbrace{\frac{-t^2}{s+a} e^{-(s+a)t} \Big|_0^\infty}_{=0,\ Re[s] > -a} + \frac{1}{s+a} \int_0^\infty 2t\, e^{-(s+a)t}\, dt$$

The second term above is known from part (1) above. Therefore,

$$\boxed{G(s) = \frac{2}{(s+a)^3}, \quad Re[s] > -a}$$

3. The pattern should be clear. In general,

$$\boxed{\mathcal{L}[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}}, \quad Re[s] > -a}$$

4. For

$$f(t) = e^{-(t-a)^2/2b^2}, \qquad \text{for all } t$$

the LT is

$$F(s) = \int_{-\infty}^{\infty} e^{-(t-a)^2/2b^2} e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-(t-a)^2/2b^2 - st}\, dt$$

The exponent is

$$-\frac{(t-a)^2}{2b^2} - st = -\frac{t^2}{2b^2} + \frac{at}{b^2} - \frac{a^2}{2b^2} - st$$

$$= -\frac{t^2}{2b^2} + \left(\frac{a}{b^2} - s\right)t - \frac{a^2}{2b^2}$$

Complete the square to obtain

$$\text{exponent} = -\frac{1}{2b^2}\left(t - [a - sb^2]\right)^2 + \frac{s^2b^2}{2} - as$$

Therefore,

$$F(s) = \int_{-\infty}^{\infty} e^{-(t-(a-sb^2))^2/2b^2} e^{s^2b^2/2 - as}\, dt$$

$$G(s) = e^{s^2b^2/2 - as} \int_{-\infty}^{\infty} e^{-[t-(a-sb^2)]^2/2b^2}$$

The integral above has integrand which is a Gaussian. Therefore,

$$G(s) = e^{s^2b^2/2 - as} \cdot \underbrace{\sqrt{2\pi} \cdot b}_{\text{Factors required to normalize Gaussian}}$$

The integral converges for all s, because the exponent is dominated by $-t^2/2b^2$. Therefore,

$$\boxed{F(s) = \sqrt{2\pi}\, b\, e^{s^2b^2/2 - as}, \quad \text{all } s}$$

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Engineer's Computation Pad
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