F3 – Lecture Notes

1. Thin-Airfoil Analysis Problem (continued)

Reading: Anderson 4.8

Cambered airfoil case

We now consider the case where the camberline $Z(x)$ is nonzero. The general thin airfoil equation, which is a statement of flow tangency on the camberline, was derived previously.

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\ \sin\theta\, d\theta}{\cos\theta - \cos\theta_o} \;=\; V_\infty\left(\alpha - \frac{dZ}{dx}\right) \tag{1}$$

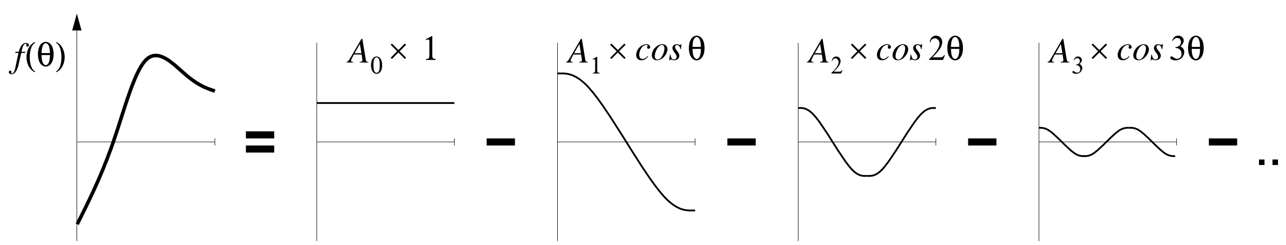
For an arbitrary camberline shape $Z(x)$, the slope dZ/dx varies along the chord, and in the equation it is negated and shifted by the constant α. Let us consider this combination to be some general function of θ_o.

$$\alpha - \frac{dZ}{dx} \;\equiv\; f(\theta_o)$$

For the purpose of computation, any such function can be conveniently represented or approximated by a *Fourier cosine series* ,

$$f(\theta_o) \;=\; A_0 \;-\; \sum_{n=1}^{N} A_n\ \cos n\theta_o$$

which is illustrated in the figure. The negative sign in front of the sum could be absorbed into all the coefficients, but is left outside for later algebraic simplicity.



The overall summation can be made arbitrarily close to a known $f(\theta_o)$ by making N sufficiently large (i.e. using sufficiently many terms). The required coefficients A_0, A_1, $\ldots A_N$ are computed one by one using *Fourier analysis* , which is the evaluation of the following integrals.

$$\begin{aligned}
A_0 &= \frac{1}{\pi}\int_0^\pi f(\theta)\, d\theta \\
A_1 &= \frac{2}{\pi}\int_0^\pi f(\theta)\ \cos\theta\, d\theta \\
A_2 &= \frac{2}{\pi}\int_0^\pi f(\theta)\ \cos 2\theta\, d\theta \\
&\vdots \\
A_N &= \frac{2}{\pi}\int_0^\pi f(\theta)\ \cos N\theta\, d\theta
\end{aligned}$$

For the particular $f(\theta_o)$ used here, these integrals become

$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dZ}{dx}\, d\theta$$

$$A_n = \frac{2}{\pi}\int_0^\pi \frac{dZ}{dx}\, \cos n\theta \, d\theta \qquad\qquad (n = 1, 2, \ldots)$$

In practice, the integrals can be evaluated either analytically or numerically. If dZ/dx is smooth, then the higher A_n coefficients will rapidly decrease, and at some point the remainder can be discarded (the series truncated) with little loss of accuracy.

Replacing $\alpha - dZ/dx$ in equation (1) with its Fourier series gives the integral equation

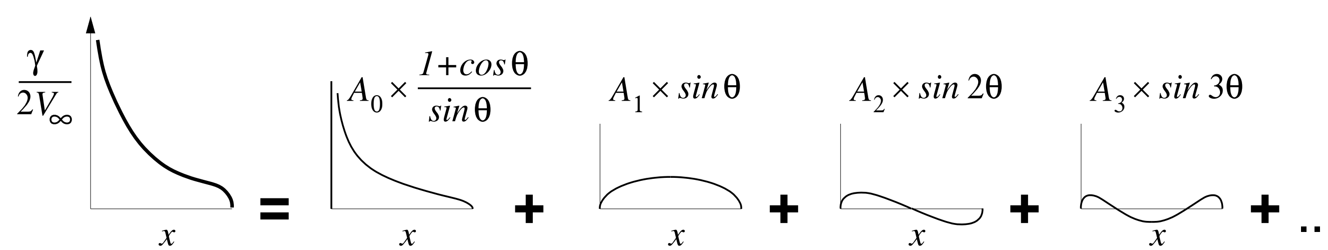
$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\, \sin\theta \, d\theta}{\cos\theta - \cos\theta_o} = V_\infty \left(A_0 - \sum_{n=1}^{N} A_n \cos n\theta_o \right) \qquad (2)$$

which is to be solved for the unknown $\gamma(\theta)$ distribution. As before, the solution of this integral equation is beyond scope here. Again, let us simply state the solution.

$$\gamma(\theta) = 2V_\infty \left(A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{N} A_n \sin n\theta \right)$$

The leading term is the same as for the zero-camber case, but with A_0 replacing α. The remaining coefficients $A_1, A_2, \ldots$ in the summation depend only on the shape of the camberline, and in particular are independent of α.

The figure shows the contributions of the various terms towards γ, all plotted versus the physical x coordinate rather than versus θ. Note that here the coefficients A_0, $A_1 \ldots A_N$ have



already been determined, and are now merely used to construct $\gamma(\theta)$ by simple summation of the series. This $\gamma(\theta)$ will now be integrated to obtain the lift force and moment.

Force calculation

The circulation and lift/span are computed in the same manner as with the symmetric airfoil case.

$$\Gamma = \int_0^c \gamma(\xi)\, d\xi \quad , \qquad L' = \rho V_\infty \Gamma$$

The integral is again most easily performed in the trigonometric coordinate θ.

$$\Gamma = \frac{c}{2}\int_0^\pi \gamma(\theta)\, \sin\theta \, d\theta = cV_\infty \left[A_0 \int_0^\pi (1+\cos\theta)\, d\theta + \sum_{n=1}^{N} A_n \int_0^\pi \sin n\theta \, \sin\theta \, d\theta \right]$$

The first integral in the brackets is easily evaluated.

$$\int_0^\pi (1+\cos\theta)\, d\theta = \pi$$

The integrals inside the summation can be evaluated by using the *orthogonality property* of the sine functions.

$$\int_0^\pi \sin n\theta \, \sin m\theta \, d\theta \;=\; \left\{ \begin{array}{ll} \pi/2 & \text{(if } n = m) \\ 0 & \text{(if } n \neq m) \end{array} \right.$$

We see that only the $n = 1$ integral inside the summation evaluates to $\pi/2$, and all the others are zero. The final result is

$$\begin{aligned} \Gamma &= c\,V_\infty \left(\pi A_0 \,+\, \frac{\pi}{2} A_1\right) \\ L' &= \rho V_\infty \Gamma \;=\; \rho V_\infty^2 c\,\pi \left(A_0 \,+\, \frac{1}{2} A_1\right) \\ c_\ell &= \frac{L'}{\frac{1}{2}\rho V_\infty^2 c} \;=\; \pi\,(2A_0 \,+\, A_1) \end{aligned}$$

It's informative to substitute the previously-obtained expressions for A_0 and A_1, giving

$$c_\ell \;=\; 2\pi \left[\alpha \;-\; \frac{1}{\pi}\int_0^c \frac{dZ}{dx}\,(1 - \cos\theta_o)\; d\theta_o\right]$$

The integral term inside the brackets depends only on the camberline shape, and is independent of the angle of attack. Hence the lift slope is

$$\frac{dc_\ell}{d\alpha} \;=\; 2\pi$$

which is the same as for the symmetrical airfoil case. We therefore reach the important conlcusion that camber has no influence on the lift slope. A terse and convenient way to represent the $c_l(\alpha)$ function is therefore

$$c_\ell \;=\; \frac{dc_\ell}{d\alpha}\,(\alpha - \alpha_{L=0})$$

where $\alpha_{L=0}$ is called the *zero-lift angle* , which depends only on the camberline shape.

$$\alpha_{L=0} \;=\; \frac{1}{\pi}\int_0^c \frac{dZ}{dx}\,(1 - \cos\theta_o)\; d\theta_o$$

The moment/span about the leading edge is again computed using the trigonometric coordinate.

$$M'_{\mathrm{LE}} \;=\; -\rho V_\infty \int_0^c \gamma\,\xi d\xi \;=\; -\rho V_\infty \frac{c^2}{4}\int_0^c \gamma(\theta)\,(1{-}\cos\theta)\,\sin\theta d\theta \;=\; -\rho V_\infty^2 \frac{c^2}{4}\pi\Big(A_0 + A_1 - \frac{1}{2}A_2\Big)$$

The moment/span and corresponding moment coefficient about the $x = c/4$ quarter-chord point are

$$\begin{aligned} M'_{c/4} &= M'_{\mathrm{LE}} \,+\, \frac{c}{4}L' \;=\; \rho V_\infty^2 \frac{c^2}{4}\,\frac{\pi}{2}(A_2 - A_1) \\ c_{m,c/4} &= \frac{M'_{c/4}}{\frac{1}{2}\rho V_\infty^2 c^2} \;=\; \frac{\pi}{4}\,(A_2 \,-\, A_1) \end{aligned}$$

An important result is that this $c_{m,c/4}$ depends only on the camberline shape, but not on the angle of attack. Therefore, the quarter-chord location is the *aerodynamic center* for any airfoil, defined as the location about which the moment is independent of α, or

$$\frac{dc_{m,c/4}}{d\alpha} = 0$$

Summary

For airfoil analysis, Thin Airfoil Theory takes in the following inputs:

α	angle of attack
dZ/dx	camberline slope distribution along chord

The outputs are:

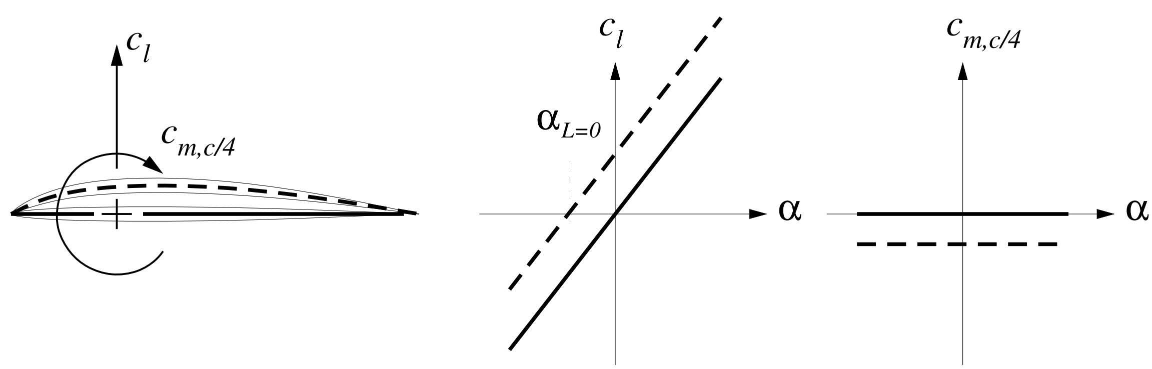
c_ℓ	lift coefficient
c_m	moment coefficient, about $c/4$ or any other location

The information propagates as follows.

$$\alpha\,,\ \frac{dZ}{dx}(\theta_o) \quad \xrightarrow{\text{Fourier analysis}} \quad A_0\,,\ A_1 \ldots A_N \quad \xrightarrow{\text{series summing}} \quad \gamma(\theta) \quad \xrightarrow{\text{chordwise integration}} \quad c_\ell\,,\ c_m$$

The Fourier coefficients A_n and the vortex sheet strength distribution $\gamma(\theta)$ are intermediate results.

The influence of camber on the airfoil $c_\ell(\alpha)$ and $c_{m,c/4}(\alpha)$ curves is illustrated in the figure.



These results are subject to the assumptions inherent in thin airfoil theory. In practice, they are surprisingly accurate even for relatively thick or highly-cambered airfoils. It appears to be better at predicting trends (with camber, α, etc) than absolute numbers. When used merely as a conceptual framework for understanding airfoil behavior rather than for quantitative predictions, thin airfoil theory is highly applicable to almost any airfoil.