# F4 – Lecture Notes

1. Dimensional Analysis – Buckingham Pi Theorem
2. Dynamic Similarity – Mach and Reynolds Numbers

Reading: Anderson 1.7

## Dimensional Analysis

### Physical parameters

A large number of physical parameters determine aerodynamic forces and moments. The following parameters are involved in the production of lift.

| Parameter | Symbol | Units |
|---|---|---|
| Lift per span | $L'$ | $mt^{-2}$ |
| Angle of attack | $\alpha$ | — |
| Freestream velocity | $V_\infty$ | $lt^{-1}$ |
| Freestream density | $\rho_\infty$ | $ml^{-3}$ |
| Freestream viscosity | $\mu_\infty$ | $ml^{-1}t^{-1}$ |
| Freestream speed of sound | $a_\infty$ | $lt^{-1}$ |
| Size of body (e.g. chord) | $c$ | $l$ |

For an airfoil of a given shape, the lift per span in general will be a function of the remaining parameters in the above list.

$$\begin{aligned} L' &= f(\alpha, \rho_\infty, V_\infty, c, \mu_\infty, a_\infty) \\ \text{or} \qquad f - L' &= g(L', \alpha, \rho_\infty, V_\infty, c, \mu_\infty, a_\infty) \;=\; 0 \end{aligned}$$

In this particular example, the $g$ function has $N = 7$ parameters, expressed in a total of $K = 3$ units (mass $m$, length $l$, and time $t$).

### Dimensionless Forms

The Buckingham Pi Theorem states that this function can be rescaled into an equivalent *dimensionless* function

$$\bar{g}(\,\Pi_1, \Pi_2 \ldots \Pi_{N-K}) \;=\; 0$$

with unly $N - K = 4$ dimensionless parameters, or Pi products. These are products of the parameters of the dimensional function $g$. In this particular case, the Pi products are:

$$\begin{aligned} \Pi_1 &= \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2\, c} &&= c_\ell && \text{lift coefficient} \\ \Pi_2 &= \alpha &&= \alpha && \text{angle of attack} \\ \Pi_3 &= \frac{\rho_\infty V_\infty\, c}{\mu_\infty} &&= Re && \text{Reynolds number} \\ \Pi_4 &= \frac{V_\infty}{a_\infty} &&= M_\infty && \text{Mach number} \end{aligned}$$

The $\bar{g}$ function is therefore

$$\bar{g}(c_\ell, \alpha, Re, M_\infty) \;=\; 0$$

which can be considered to give $c_\ell$ in terms of the remaining three parameters.

$$c_\ell \;=\; \bar{f}(\alpha, Re, M_\infty)$$

**Derivation of dimensionless forms**

Anderson 1.7 has details on how the Pi product combinations can be derived for any complex situation using linear algebra. In many cases, however, the products can be obtained by physical insight, or perhaps by inspection. Several rules can be applied here:

- Any parameter which is already dimensionless, such as $\alpha$, is automatically one of the Pi products.
- If two parameters have the same units, such as $V_\infty$ and $a_\infty$, then their ratio ($M_\infty$ in this case) will be one of the Pi products.
- A power or simple multiple of a Pi product is an acceptable alternative Pi product. For example, $(V_\infty/a_\infty)^2$ is an acceptable alternative to $V_\infty/a_\infty$, and $\rho_\infty V_\infty^2$ is an acceptable alternative to $\frac{1}{2}\rho_\infty V_\infty^2$. Which particular forms are used is a matter of convention.
- Combinations of Pi products can replace the originals. For example, the 3rd and 4th products in the example could have been defined as

$$\Pi_3 \;=\; \frac{\rho_\infty a_\infty\, c}{\mu_\infty} \;=\; Re/M_\infty$$
$$\Pi_4 \;=\; \frac{V_\infty}{a_\infty} \;=\; M_\infty$$

  which is workable alternative, but perhaps less practical, and certainly less traditional.

## Dynamic Similarity

It is quite possible for two differently-sized physical situations, with different dimensional parameters, to nevertheless reduce to the same dimensionless description. The only requirement is that the corresponding Pi products have the same numerical values.

**Airfoil flow example**

Consider two airfoils which have the same shape and angle of attack, but have different sizes and are operating in two different fluids. Let's omit the $()_\infty$ subscript for clarity.

| Airfoil 1 (sea level) | Airfoil 2 (cryogenic tunnel) |
|---|---|
| $\alpha_1 = 5°$ | $\alpha_2 = 5°$ |
| $V_1 = 210\text{m/s}$ | $V_2 = 140\text{m/s}$ |
| $\rho_1 = 1.2\text{kg/m}^3$ | $\rho_2 = 3.0\text{kg/m}^3$ |
| $\mu_1 = 1.8 \times 10^{-5}\text{kg/m-s}$ | $\mu_2 = 1.5 \times 10^{-5}\text{kg/m-s}$ |
| $a_1 = 300\text{m/s}$ | $a_2 = 200\text{m/s}$ |
| $c_1 = 1.0\text{m}$ | $c_2 = 0.5\text{m}$ |

The Pi products evaluate to the following values.

| Airfoil 1 | Airfoil 2 |
|---|---|
| $\alpha_1 = 5°$ | $\alpha_2 = 5°$ |
| $Re_1 = 1.4 \times 10^7$ | $Re_2 = 1.4 \times 10^7$ |
| $M_1 = 0.7$ | $M_2 = 0.7$ |

Since these are also the arguments to the $\bar{f}$ function, we conclude that the $c_\ell$ values will be the same as well.

$$\begin{aligned}\bar{f}(\alpha_1, Re_1, M_1) &= \bar{f}(\alpha_2, Re_2, M_2) \\ c_{\ell_1} &= c_{\ell_2}\end{aligned}$$

When the nondimensionalized parameters are equal like this, the two situations are said to have *dynamic similarity*. One can then conclude that any other dimensionless quantity must also match between the two situations. This is the basis of wind tunnel testing, where the flow about a model object duplicates and can be used to predict the flow about the full-size object. The prediction is correct only if the model and full-size objects have dynamic similarity.

**Approximate dynamic similarity**

Frequently it is not essential to exactly match all the dimensionless parameters to obtain a good correspondence between two flows. For example, for low speed flows where $M_\infty < 0.3$, the precise value of this Mach number has little effect on the flow. Likewise, the Reynolds number has little effect on the lift provided $Re > 10^6$ or so. If these conditions are met, then we can assume that

$$c_\ell = \bar{f}(\alpha)$$

and using wind tunnel data to predict lift simply requires that the angle of attack be matched. This may not be true of the other aerodynamic force coefficients. The $c_d$ of an airfoil, for example, is likely to always have a significant dependence on the Reynolds number which cannot be neglected.

$$c_d = \bar{f}(\alpha, Re)$$