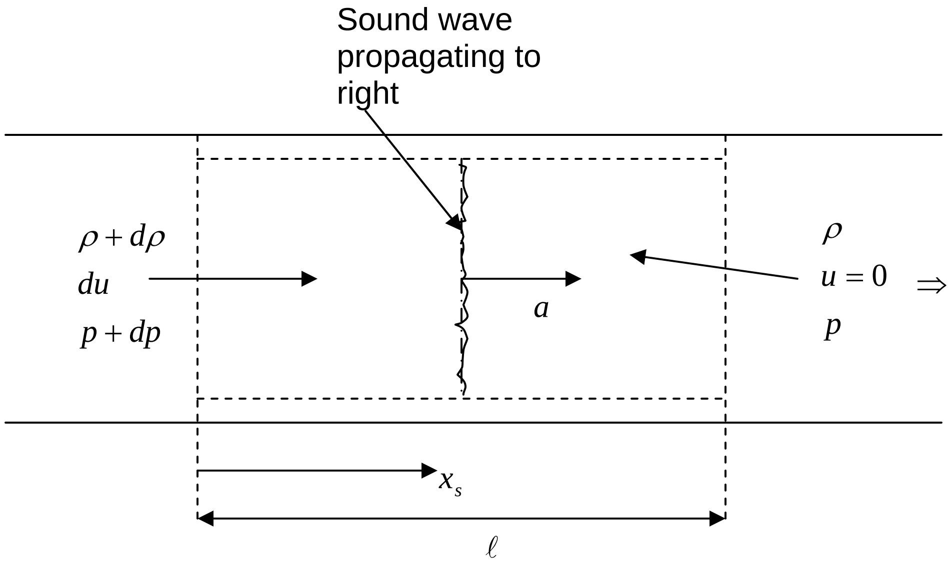
Derivation of Sound Wave Properties



$$\Rightarrow \rho(x) = \begin{cases} \rho + d\rho, & x < x_s \\ \rho, & x > x_s \end{cases}$$

Similar for $u(x), p(x)$

<u>Assume</u>:

∗ Sound wave creates small disturbances in an isentropic manner.

$\boxed{\text{Mass}}$

$$\frac{d}{dt}\int_0^\ell \rho(x)dx + \rho u\Big|_\ell - \rho u\Big|_0 = 0$$

$$\frac{d}{dt}\left[\int_0^{x_s}(\rho + d\rho)dx + \int_{x_s}^\ell \rho dx\right] + \underbrace{\cancel{\rho u\Big|_\ell}}_{0} - (\rho + d\rho)du = 0$$

$$\frac{d}{dt}\left[\int_0^{x_s} d\rho dx + \underbrace{\int_0^\ell \rho dx}_{\text{Constant time}}\right] - (\rho + d\rho)du = 0$$

$$\frac{d}{dt}\left[d\rho x_s\right] - (\rho + d\rho)du = 0$$

$$d\rho\frac{dx_s}{dt} - (\rho + d\rho)du = 0$$

$$d\rho\ a - \rho du - \underbrace{d\rho du}_{\text{Higher order}} = 0$$

$$\Rightarrow \boxed{d\rho a = \rho du}$$

Isentropic disturbances

Constant entropy disturbances for perfect, ideal gases satisfy:

$$\frac{p}{\rho^\gamma} = const.$$

$$\Rightarrow \frac{p+dp}{(\rho+d\rho)^\gamma} = \frac{p}{\rho^\gamma}$$

$$\Rightarrow \frac{p+dp}{p} = \left(\frac{\rho+d\rho}{\rho}\right)^\gamma$$

$$1+\frac{dp}{p} = \left(1+\frac{d\rho}{\rho}\right)^\gamma$$

$$1+\frac{dp}{p} = 1+\gamma\frac{d\rho}{\rho}$$

$$\boxed{dp = \frac{\gamma p}{\rho} d\rho}$$

Conservation of Momentum

$$\underbrace{\frac{d}{dt}\int_0^\ell \rho u dx}_{\rho du a} + \underbrace{(\rho u^2 + p)\Big|_\ell}_{+\ p} - \underbrace{(\rho u^2 + p)\Big|_0}_{-\ (p+dp)} = 0$$

$$\rho du a + p - (p+dp) = 0$$

Higher order terms eliminated

$$\Rightarrow \boxed{\rho a du = dp}$$

Summarizing:

Mass: $a d\rho = \rho du$ (1)

Isentropic: $dp = \frac{\gamma p}{\rho} d\rho$ (2)

Momentum: $\rho a du = dp$ (3)

Combining $a*(1)-(3)$ gives:

$$a^2 d\rho = dp$$

Then, using (2) gives:

$$a^2 d\rho = \frac{\gamma p}{\rho} d\rho$$

$$\Rightarrow \left(a^2 - \frac{\gamma p}{\rho}\right) d\rho = 0$$

Since $d\rho \neq 0$, then $\boxed{a^2 = \frac{\gamma p}{\rho}}$. We've just derived the speed of sound for an

Ideal, perfect gas.

<u>Note</u>: without assuming ideal, perfect gas, the general result is $a^2 = \frac{\partial p}{\partial \rho}\Big|_{s=const.}$

One other thing of interest: Suppose the sound wave caused a change in pressure, dp. Then, the change in velocity is:

$du = \frac{1}{\rho a} dp \Rightarrow du$ has same sign as dp