Lecture 11 Sept 29, 03.

SOLUTION TECHNIQUES

4 a) Numerical Methods for ODE.

A) Reduction to 1st order system

B) Discretization

C) Stability

Reading: Numerical Comp. of Int. & Ext. Flows Vol I C. Hirsch. 267-290
ATP 76-83.

A) An N^{th} order ODE can always be reduced to n 1st order ODEs.

$$y^{(n)} = F(t, y, y', \cdots y^{(n-1)})$$

Define $x_1 = y, \; x_2 = y', \; \cdots\cdots, x_n = y^{(n-1)}$

$$\Rightarrow \quad \begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \end{aligned}$$

$$x_n' = F(t, x_1, x_2, \cdots, x_n)$$

Example 1) Falkner-Skan Eqn for $F(\eta)$

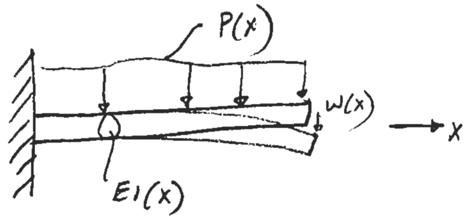
$$F''' + \frac{\beta_u+1}{2} F F'' + \beta_u (1 - F'^2) = 0$$

$$F' = U$$
$$U' = S$$
$$S' = -\frac{\beta_u+1}{2} F S - \beta_u (1 - U^2)$$

or $\begin{bmatrix} F \\ U \\ S \end{bmatrix}' = f\left[\begin{pmatrix} F \\ U \\ S \end{pmatrix}; \beta\right]$

2> Beam Equation

$$\left(EI(x)\,w''\right)'' = p(x)$$



① $w' = t$

$t' = u$

$u' = v$

$$\Rightarrow \begin{Bmatrix} w \\ t \\ u \\ v \end{Bmatrix}' = \begin{bmatrix} \\ & A & \\ \\ \end{bmatrix} \begin{Bmatrix} w \\ t \\ u \\ v \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ p/EI \end{Bmatrix}$$

$EI\,v' + 2EI'\,v + EI''\,u = p$

$$\left[[EI\,u]'\right]' = \left[(EI)'u + \overset{v}{\cancel{u'}}\,EI\right]'$$

② Alternatively

$w' = t$

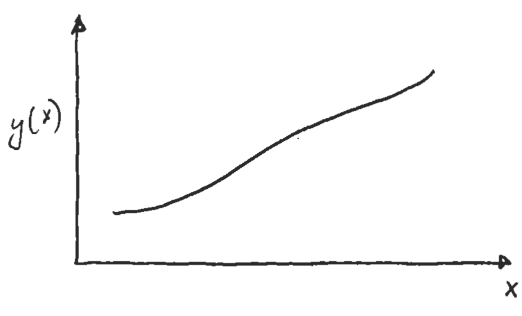
$EI t' = u$ ← change, no need to differentiate EI illustrates alt. approach to order reduction

$u' = v$

$v' = p$

We need to examine how to solve 1st order ODE → $y' = f(x, y(x))$

B> Discretization



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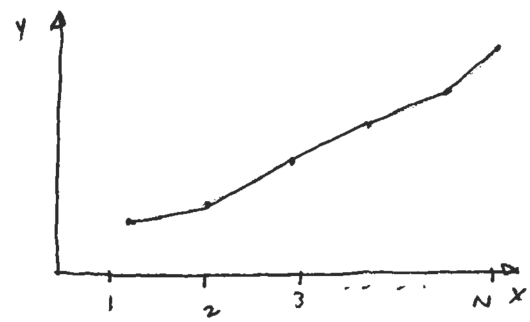


Cont. System

① $y(x)$ governed by ODE and ICs and/or B.Cs

② ∞ DOF

Discrete System $1 \le i \le N$

y_i governed by N algebraic equations. (including I.C, BC)

N DOFs

Example

$$y' = -\alpha y \quad ; \quad \alpha \geq 0$$

Exact Solution: $y = y_0 e^{-\alpha x}$

Discretize using forward Euler

$$y_{i+1} = y_i + \Delta x y_i' = y_i - \Delta x \alpha y_i$$
$$= y_i (1 - \Delta x \alpha)$$

$$y_{i+2} = y_{i+1} - \Delta x \alpha y_{i+1}$$
$$= y_{i+1}(1 - \Delta x \alpha) = y_i (1 - \Delta x \alpha)^2$$

in general $\therefore \; y_n = y_0 (1 - \alpha \Delta x)^n$

The discretization is <u>consistent</u> if $y_i \to y_{exact}$ as $\Delta x \to 0$

$$\lim_{n \to \infty} y_0 \left(1 - \frac{\alpha X}{n}\right)^n \quad \text{where } X = \underset{\substack{\uparrow \\ \text{length of domain}}}{X_{end} - X_0}$$

$$= y_0 e^{-\alpha X}$$ $\therefore$ discretization is consistent

c) <u>Stability</u>

Discretization is stable if error between y_i & y_{exact} stays bound as $n \to \infty$.

$$y_n = y_0 (1 - \alpha \Delta x)^n$$

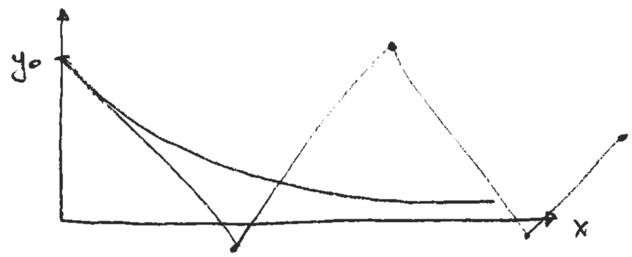
$$\left|\frac{y_n}{y_0}\right| = |1 - \alpha \Delta x|^n \qquad \alpha > 0$$

For $\alpha \geq 0$ $\quad y_{exact} = y_0 e^{-\alpha X}$ decays as $x \to \infty$

$\therefore$ for $|y_n / y_0|$ to decay $\quad |1 - \alpha \Delta x| \leq 1 \quad (|1 - \alpha \Delta x| > 1, y_n \to \text{grows})$

$\therefore \; |1 - \alpha \Delta x| \leq 1$ is a stability requirement

$\Rightarrow \quad \alpha \Delta x \leq 2 \quad$ for stability



Backward Euler Discretization

$$y' = -\alpha y$$

$$y_{i+1} = y_i + \Delta x \, y'_{i+1} = y_i - \Delta x \, \alpha \, y_{i+1}$$

$$= \frac{y_i}{1 + \Delta x \alpha}$$

$$\therefore \; y_n = y_0 (1 + \alpha \Delta x)^{-n}$$

$\therefore \quad \alpha \geq 0 \quad$ stable if $\quad |1 + \alpha \Delta x| \geq 1 \quad \Rightarrow \quad$ stable for all Δx

a) For simple 1-equation systems, stability and accuracy requirements on Δx are typically the same

b) For multi-equation systems, they can be very different.

$$\epsilon y'' + y' - y = 0 \qquad \begin{array}{l} y(0) = 1 \\ y'(0) = 0 \end{array}$$

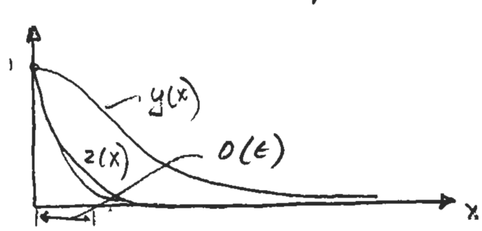
Reduce order to 1st

$$z = y' - y$$

$$\therefore \; y' = -y + z \qquad y(0) = 1$$

$$z' = -z/\epsilon \qquad z(0) = 1$$

<u>Exact solution</u>



$$\vec{y}' = -[\alpha]\vec{y}$$

$$y_{n+1} = [I - \alpha \Delta x]\, y_n$$

$$[A] \rightarrow 0 \qquad \text{as } n \rightarrow \infty$$

if the largest eigenvalue of $[A]$ has magnitude less than 1. Thus y_n is bounded, or

$$\Delta x < 2/\lambda_{max}$$

Using Forward Euler

$$y_{i+1} = y_i + \Delta x(-y_i + z_i) = y_i(1 - \Delta x) + z_i \Delta x$$

$\alpha = 1$

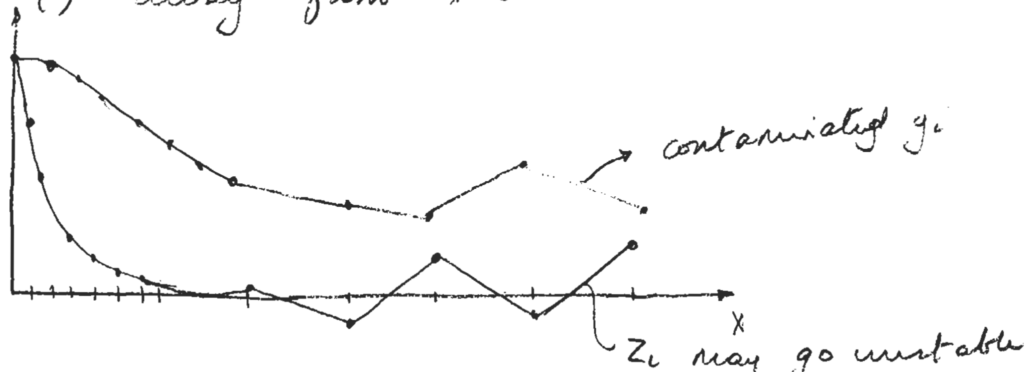
$\Delta x = 1.1$

$n = 5$

$$z_{i+1} = z_i - z_i/\epsilon \, \Delta x = z_i\left(1 - \frac{\Delta x}{\epsilon}\right)$$

$$\begin{Bmatrix} y_{i+1} \\ z_{i+1} \end{Bmatrix} = \begin{bmatrix} 1 - \Delta x & \Delta x \\ 0 & \left(1 - \frac{\Delta x}{\epsilon}\right) \end{bmatrix} \begin{Bmatrix} y_i \\ z_i \end{Bmatrix}.$$

To accurately resolve z close to $x = 0$ we need $\Delta x \sim O(\epsilon)$, and $\Delta x \sim O(1)$ away from $x = 0$



spacing dictated by accuracy may result in instability even after $z \to 0$. Excess work is done in maintaing small $\Delta x \sim O(\epsilon)$ (excess work $\sim 1/\epsilon$)

Instability in z_i contaminates y_i. Alternative is to use Backward Euler Scheme

$$\Delta x_{\text{stability}} \ll \Delta x_{\text{accuracy}} \rightarrow \text{stiff ODE system}$$

Stiffness occurs when there are 2 or more different scales of the independant variable.

$y' = z - y$ $\quad$ $y'' = z' - y'$

$\epsilon z' = -z$

$\epsilon(y'' + y') = -y' - y$

$\epsilon y'' + y'($

[stability + stiffness are 2 imp issues for solving TSL eqns numerically]

Stability

$$y_{n+1} = y_n(1-\alpha\Delta X)$$
$$y_n = y_0(1-\alpha\Delta X)^n$$

Error

$$y_n = y_{exact_n} + \epsilon_n$$

Substituting above gives

$$\epsilon_{n+1} = \epsilon_n(1-\alpha\Delta X)$$

(since $y_{n\,exact}$ satisfies the difference eqn.)

For the Forward Euler scheme to be stable

$\left|\frac{\epsilon_{n+1}}{\epsilon_n}\right|$ must not grow $\therefore \leq 1$

Assume $\quad \epsilon_n = Y_n e^{in\phi}$

$$Y_{n+1} e^{i(n+1)\phi} = Y_n e^{in\phi}(1-\alpha\Delta X)$$

or $\quad \dfrac{Y_{n+1}}{Y_n} = \underbrace{(1-\alpha\Delta X)e^{-i\phi}}$ amplification factor

$$\left|\frac{Y^{n+1}}{Y^n}\right|^2 = (1-\alpha\Delta X)^2$$

$$\Rightarrow |1-\alpha\Delta X| \leq 1 \Rightarrow \alpha\Delta X \leq 2 \;//$$

Stiff system

$$\epsilon y'' + y'(1+\epsilon) + y = 0 \qquad y(0)=1, \quad y'(0)=0$$

$$z = y'+y, \quad z' = y''+y'$$

$$\Rightarrow \epsilon(y''+y') + (y'+y) = 0$$

$$\epsilon z' + z = 0 \Rightarrow z' = -z/\epsilon$$

Equivalent system:

$$y' = -y+z$$
$$z' = -z/\epsilon \quad //$$

Stability on z $\quad \left|\frac{\Delta X}{\epsilon}\right| \leq 2. \qquad \Delta X \leq 2\epsilon$

$y' = -y+z$
$z' = -z/\epsilon$
$y'' = -y'+z'$
$= -y' - z/\epsilon$
$= -y' - \dfrac{y-y'}{\epsilon}$
$\epsilon y'' = -y'\epsilon - y - y'$
$= -y'(1+\epsilon) - y$
$\boxed{\epsilon y'' + y'(1+\epsilon) + y = 0}$
$z = y+y'$