Thin Shear Layer Approx.

3.4>

A> Local Scaling

B> Falkner-Skan flows.

Reading:

Betch 308 - 314 White 233 - 246

Sch 201 - 206

- 2 Handouts : Shear Layer B.Cs , Local scaling, TSL - graphical description
- collect PS 2 , handout PS 3

Similarity Solution:

A> Reducing the # of independant variables by one or more by some analytical means (exploit)

Example: $y, t \longrightarrow \eta = y/\Delta(t)$ (Rayleigh problem)

PDE $\longrightarrow$ ODE.

$u(y,t) \longrightarrow u(\eta)$

In the case of a wall boundary layer we seek similarity

$$x, y \rightarrow \xi ; \eta$$
$$u(x,y),\ \psi(x,y) \longrightarrow u(\eta),\ \psi(\eta)$$

— x —

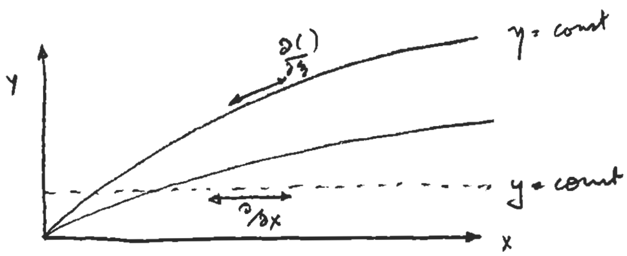
In general, the TSL will increase in thickness downstream which suggests using <u>local</u> length and velocity scales to normalize y, ψ, u, etc.

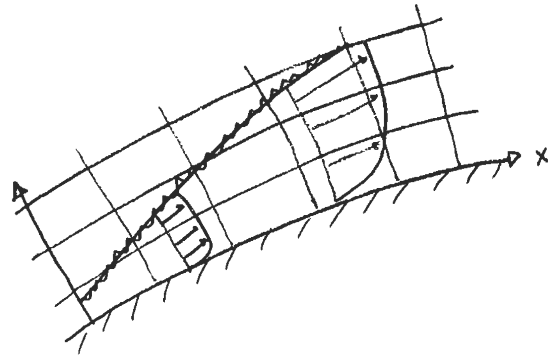
. Assume $\Delta(x)$ is some transverse length scale of $O(\delta)$

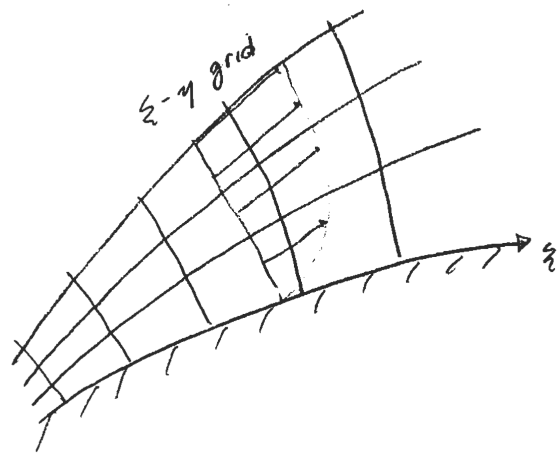
$$\eta = y/\Delta(x) \qquad \Delta(x) = O(\delta)$$
$$\xi = x$$

$$\Rightarrow \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} - \frac{\eta}{\Delta}\cdot\frac{d\Delta}{d\xi}\cdot\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{1}{\Delta}\frac{\partial}{\partial \eta}$$







We must also transform ψ, u, τ using local scales

$\psi = F(\xi,\eta)\cdot m$ $\qquad m = u_e \Delta$ (mass flow scale)

$u = u_e\, U(\xi,\eta)$

$\tau = S(\xi,\eta)\cdot\frac{\Delta}{\xi}\cdot\rho u_e^2$

$\psi - \psi_0 = O(u_e\Delta)$

Volume flow $= \psi_2 - \psi_1$

TSL Eqn:

$$\frac{\partial \psi}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \frac{\partial(\tau/\rho)}{\partial y}$$

Let's (transform) examine convective terms

$$\frac{\partial \psi}{\partial y} = \frac{1}{\Delta}\frac{\partial}{\partial \eta}(F u_e \Delta) = u_e \frac{\partial F}{\partial \eta}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \xi}(u_e V) - \frac{\partial}{\partial \eta}(u_e V)\cdot\frac{\eta}{\Delta}\frac{d\Delta}{d\xi}$$

$$= V\frac{du_e}{d\xi} - u_e\frac{\partial V}{\partial \eta}\cdot\frac{\eta}{\Delta}\cdot\frac{d\Delta}{d\xi} + u_e\frac{\partial V}{\partial \xi}$$

$$\Rightarrow \frac{\partial \psi}{\partial y}\cdot\frac{\partial u}{\partial x} = u_e\frac{\partial F}{\partial \eta}\left[V\frac{du_e}{d\xi} - u_e\frac{\partial V}{\partial \eta}\cdot\frac{\eta}{\Delta}\frac{d\Delta}{d\xi} + u_e\frac{\partial V}{\partial \xi}\right]$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial \xi}(F u_e \Delta) - \frac{\eta}{\Delta}\frac{d\Delta}{d\xi}\cdot\frac{\partial}{\partial \eta}(F u_e \Delta)$$

$$= F\frac{\partial}{\partial \xi}(n) - \frac{\eta}{\Delta}\frac{d\Delta}{d\xi}\cdot n\frac{\partial F}{\partial \eta} + n\frac{\partial F}{\partial \xi}$$

$$\frac{\partial u}{\partial y} = \frac{u_e}{\Delta}\cdot\frac{\partial V}{\partial \eta}$$

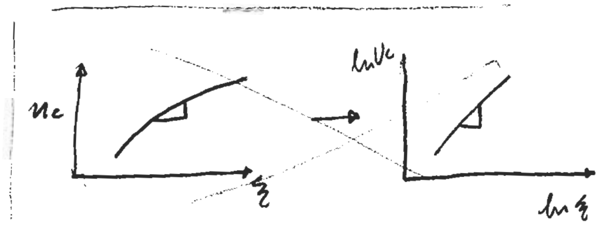
$$\Rightarrow \frac{\partial \psi}{\partial x}\cdot\frac{\partial u}{\partial y} = \frac{u_e}{\Delta}\frac{\partial V}{\partial \eta}\left[F\frac{d}{d\xi}n - \frac{\eta}{\Delta}\frac{d\Delta}{d\xi}n\frac{\partial F}{\partial \eta} + n\frac{\partial F}{\partial \xi}\right]$$

Substituting and multiplying by $\frac{\xi}{U_e^2}$ gives

$$\underset{\uparrow\,1}{\xi}\left[\frac{\partial F}{\partial \eta}\cdot\underset{\uparrow\,2}{\frac{\partial V}{\partial \xi}} - \frac{\partial F}{\partial \xi}\cdot\frac{\partial V}{\partial \eta}\right] - \underset{\uparrow\,2}{\beta_n F\frac{\partial V}{\partial \eta}} + \underset{\uparrow\,1}{\beta_u}\left(V\frac{\partial F}{\partial \eta} \underset{\uparrow\,3}{-1}\right) = \underset{\uparrow\,4}{\frac{\partial S}{\partial \eta}} \quad *$$

where,

$$\beta_u \equiv \frac{\xi}{U_e}\frac{du_e}{d\xi} = \frac{d(\ln u_e)}{d(\ln \xi)}$$



$$\beta_n \equiv \frac{\xi}{n}\frac{dn}{d\xi} = \frac{d\ln n}{d\ln \xi}$$

In order to obtain similarity, $\frac{\partial(\ )}{\partial \xi} = 0$. Require that

① $\beta_u, \beta_n = \text{const} \implies u_e \sim x^{\beta_u}$, $n \sim x^{\beta_n}$ — power law behavior

② we have $\frac{\tau}{\rho} = \nu \frac{\partial u}{\partial y} \rightarrow S = \frac{\nu \xi}{n \Delta} \frac{\partial U}{\partial \eta}$

$$\implies \frac{\nu}{n} \cdot \frac{\xi}{\Delta} = \text{const}$$

Take log of $\left(\frac{\nu}{n} \cdot \frac{\xi}{\Delta}\right)$

$$\ln \nu - \ln n + \ln \xi - \ln \Delta = \text{const.}$$

$$-d\ln n + d\ln \xi - d\ln \Delta = 0$$

$$\implies -\beta_n + 1 - \beta_\Delta = 0$$

since $n = \Delta u_e$, we have

$$\beta_u + \beta_\Delta = \beta_n$$

$$\therefore \beta_n = \frac{\beta_u + 1}{2}$$

$$\text{or } \beta_\Delta = -\tfrac{1}{2}(\beta_u - 1) \text{ — constraint on } \Delta(\xi)$$

③ In addition transformed B.C must also be independent of ξ

$y = 0 \quad U = 0, \ F = 0$

$y \to \infty \quad U = 1$

B> Falkner - Skan Transformation

If similar:

Choose

$$\Delta_{FS} = \sqrt{\frac{\nu \xi}{U_e}} \qquad \text{(from 2)}$$

$$F(\eta),\ U(\eta),\ S(\eta)$$

we have $U = F'$

$S = U'$

substituting in (*) gives

$$-\beta_n F U' + \beta_u (U F' - 1) = S'$$

Note $m = \beta_u$ (F-S notation)

$$\therefore\ S' + \frac{(m+1)}{2} F U' + m(1 - F' U) = 0$$

or

$$\boxed{F''' + \frac{m+1}{2} F F'' + m(1 - F'^2) = 0}$$

Equivalent form

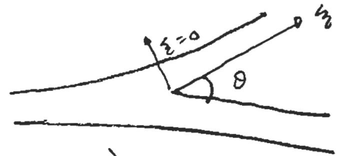
$$\tilde{\eta} = \sqrt{\frac{m+1}{2}}\,\eta \qquad \tilde{F} = \sqrt{\frac{m+1}{2}}\,F$$

$$\Rightarrow\ \tilde{F}''' + \tilde{F}\tilde{F}'' + \beta(1 - \tilde{F}'^2) = 0$$

$$\beta = \frac{\beta_u}{\beta_n} = \frac{2m}{1+m}$$

When are 1, 2, and 3 met?

- Potential wedge flows:



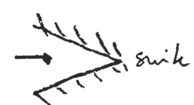
$u_e = C\xi^m$ (sheer luck) $\qquad \beta = \frac{2m}{1+m}$

$m = \frac{\theta/\pi}{2-\theta/\pi} \qquad \beta_\Delta = \frac{1-m}{2} \qquad \theta = \frac{2m}{1+m} \Rightarrow \theta = \pi\beta$

θ	m	$\Delta(\xi)$	
0	0	$C\xi^{1/2}$	— flat plate
π	1	c	— stagnation point
$\pi/2$	$1/3$	$C\xi^{1/3}$	— wedge

↑ $0 \le \beta \le 1$

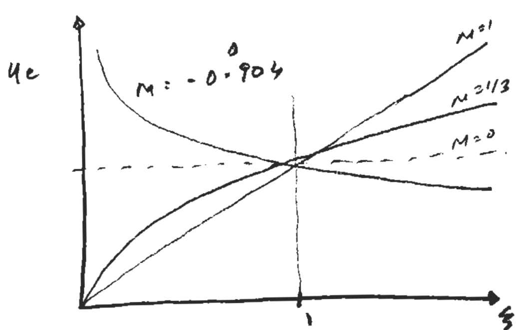
$\beta \to +\infty$ point sink



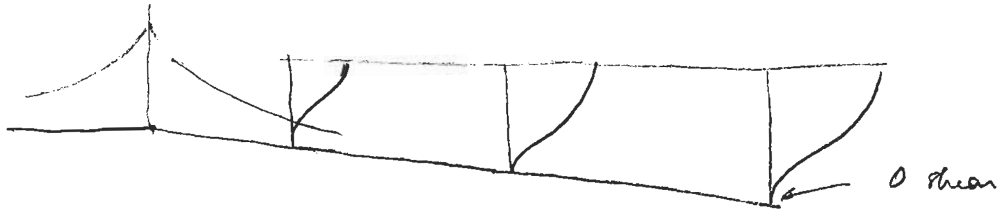
$-1 \le \beta \le 0$ expansion corner

Given some m, we can solve F.S equations $F(\eta; m)$ (one parameter family of profiles)

Handout ...



For $m = -0.0904$, $\theta \approx 0.2$ (11°)



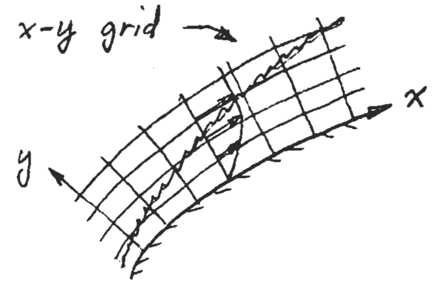
small deceleration to cause

$$\frac{u_2}{u_1} = \left(\frac{2L}{L}\right)^m \to 2^{-0.0904} \approx 0.94$$

TSL Local Scaling Transformation (Incompressible)

We wish to solve the TSL equations:

$$\boxed{\begin{aligned} u &= \frac{\partial \Psi}{\partial y} \qquad \frac{\tau}{\rho} = \nu \frac{\partial u}{\partial y} \\ \frac{\partial \Psi}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial u}{\partial y} &= u_e \frac{du_e}{dx} + \frac{\partial(\tau/\rho)}{\partial y} \end{aligned}}$$



In general, the TSL will increase in thickness downstream, making a finite-difference solution on a fixed x-y grid awkward.

Idea: Use "local" velocity and length scales, $u_e(x)$ and $\Delta(x)$, to normalize y, $\Psi(x,y)$, $u(x,y)$, etc., giving the transformation

$$(x \;\; y \;\; \Psi \;\; u \;\; \tau \;\; u_e) \rightarrow (\xi \;\; \eta \;\; F \;\; U \;\; S \;\; u_e)$$

$$\xi = x \qquad \eta = \frac{y}{\Delta} \qquad ; \quad \Delta(x) = \mathcal{O}(\text{TSL thickness})$$

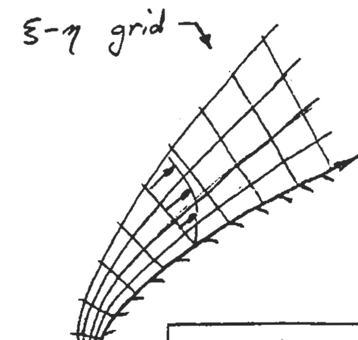
$$F = \frac{\Psi}{n} \qquad U = \frac{u}{u_e} \qquad S = \frac{\xi}{\Delta}\frac{\tau/\rho}{u_e^2} \qquad ; \quad n \equiv u_e \Delta \quad (\text{mass flow scale})$$

$$\Rightarrow \quad \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - \frac{\eta}{\Delta}\frac{d\Delta}{d\xi}\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{1}{\Delta}\frac{\partial}{\partial \eta}$$

These transformations give:

$$\boxed{\begin{aligned} U &= \frac{\partial F}{\partial \eta} \qquad S = \frac{\nu}{n}\frac{\xi}{\Delta}\frac{\partial U}{\partial \eta} \\ \frac{\partial S}{\partial \eta} + \beta_n F \frac{\partial U}{\partial \eta} + \beta_u\left(1 - U\frac{\partial F}{\partial \eta}\right) &= \xi\left(\frac{\partial F}{\partial \eta}\frac{\partial U}{\partial \xi} - \frac{\partial F}{\partial \xi}\frac{\partial U}{\partial \eta}\right) \\ \text{where} \quad \beta_n = \frac{\xi}{n}\frac{dn}{d\xi} &\qquad \beta_u = \frac{\xi}{u_e}\frac{du_e}{d\xi} \end{aligned}}$$



Requirements for similarity $\left(\frac{\partial}{\partial \xi} = 0\right)$:

1) β_u, β_n = constant $\Rightarrow$ $u_e \sim \xi^{\beta_u}$, $n \sim \xi^{\beta_n}$
2) $\frac{\nu \xi}{n \Delta}$ = constant $\Rightarrow$ $\beta_n = \frac{1}{2}(1+\beta_u)$
3) Transformed BC's = constant

Note: If $\Delta^2 = \frac{\nu \xi}{u_e}$ we recover the Falkner–Skan Transformation