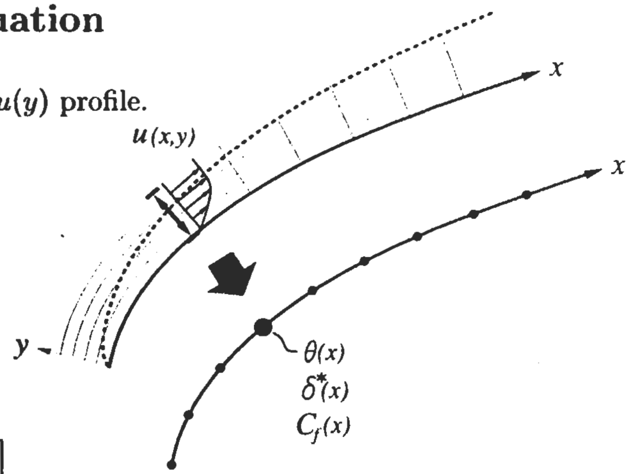
Von-Karman Integral Momentum Equation



Idea: Integrate BL flow in y, to "wash out" details in $u(y)$ profile. Converts PDEs in x, y into ODE in x.

Combine Continuity, x-momentum equations:

$$(u - u_e)\left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0\right]$$

$$+ \left[\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho_e u_e \frac{du_e}{dx} + \frac{\partial \tau}{\partial y}\right]$$

$$\Rightarrow \quad \rho u \frac{\partial u}{\partial x} + (u - u_e)\frac{\partial \rho u}{\partial x} + \rho v \frac{\partial u}{\partial y} + (u - u_e)\frac{\partial \rho v}{\partial y} = \rho_e u_e \frac{du_e}{dx} + \frac{\partial \tau}{\partial y}$$

or

$$\frac{\partial}{\partial x}[(u_e - u)\rho u] + \frac{\partial}{\partial y}[(u_e - u)\rho v] = -(\rho_e u_e - \rho u)\frac{du_e}{dx} - \frac{\partial \tau}{\partial y} \qquad (*)$$

Integrate $\int_0^{y_e} (*)\, dy$ term by term:

$$\int_0^{y_e} \frac{\partial}{\partial x}[(u_e - u)\rho u]\, dy + \int_0^{y_e} \frac{\partial}{\partial y}[(u_e - u)\rho v]\, dy = -\int_0^{y_e} (\rho_e u_e - \rho u)\frac{du_e}{dx}\, dy - \int_0^{y_e} \frac{\partial \tau}{\partial y}\, dy$$

$$\frac{d}{dx}\int_0^{y_e} [(u_e - u)\rho u]\, dy + 0 = -\frac{du_e}{dx}\int_0^{y_e} (\rho_e u_e - \rho u)\, dy + \tau_w$$

$$\boxed{\frac{d}{dx}\left(\rho_e u_e^2 \theta\right) + \rho_e u_e \delta^* \frac{du_e}{dx} = \tau_w}$$

Dimensional form

$$\boxed{\frac{d\theta}{dx} + \left(H + 2 - M_e^2\right)\frac{\theta}{u_e}\frac{du_e}{dx} = \frac{C_f}{2}}$$

Dimensionless form

Definitions

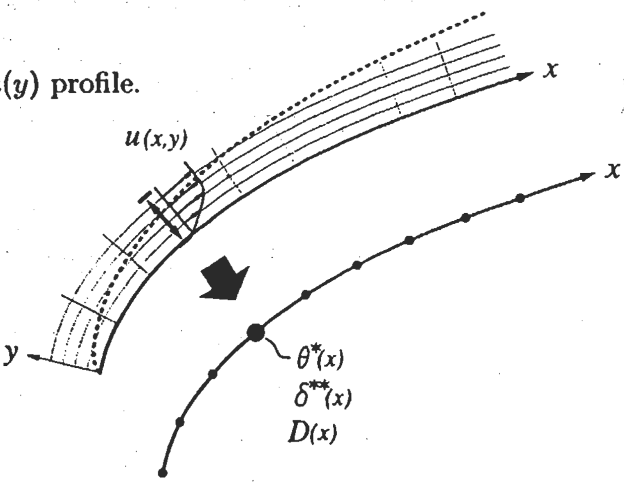
$$\theta = \int \left(1 - \frac{u}{u_e}\right)\frac{\rho u}{\rho_e u_e}\, dy$$ momentum thickness

$$\delta^* = \int \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$$ displacement thickness

$$H = \frac{\delta^*}{\theta}$$ shape parameter

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2}$$ skin friction coefficient

Integral Kinetic Energy Equation



Idea: Integrate BL flow in y, to "wash out" details in $u(y)$ profile.
Converts PDEs in x, y into ODE in x.

Combine Continuity, x-momentum equations:

$$\left(u^2 - u_e^2\right)\left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0\right]$$

$$+\ 2u\left[\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho_e u_e \frac{du_e}{dx} + \frac{\partial \tau}{\partial y}\right]$$

$$\Rightarrow \quad 2\rho u^2 \frac{\partial u}{\partial x} + (u^2 - u_e^2)\frac{\partial \rho u}{\partial x} + 2\rho u v \frac{\partial u}{\partial y} + (u^2 - u_e^2)\frac{\partial \rho v}{\partial y} = 2u\rho_e u_e \frac{du_e}{dx} + 2u\frac{\partial \tau}{\partial y}$$

or

$$\frac{\partial}{\partial x}\left[(u_e^2 - u^2)\rho u\right] + \frac{\partial}{\partial y}\left[(u_e^2 - u^2)\rho v\right] = -2(\rho_e - \rho)u u_e \frac{du_e}{dx} - 2u\frac{\partial \tau}{\partial y} \qquad (*)$$

Integrate $\int_0^{y_e} (*)\, dy$ term by term:

$$\int_0^{y_e} \frac{\partial}{\partial x}\left[(u_e^2 - u^2)\rho u\right] dy + \int_0^{y_e} \frac{\partial}{\partial y}\left[(u_e^2 - u^2)\rho v\right] dy = -\int_0^{y_e} 2(\rho_e - \rho)u u_e \frac{du_e}{dx}\, dy - \int_0^{y_e} 2u\frac{\partial \tau}{\partial y}\, dy$$

$$\frac{d}{dx}\int_0^{y_e}\left[(u_e^2 - u^2)\rho u\right] dy + 0 = -2u_e \frac{du_e}{dx}\int_0^{y_e}(\rho_e - \rho)u\, dy + 2\int_0^{y_e} \tau \frac{\partial u}{\partial y}\, dy$$

$$\boxed{\frac{d}{dx}\left(\rho_e u_e^3 \theta^*\right) + 2\rho_e u_e^2 \delta^{**}\frac{du_e}{dx} = 2D} \qquad \text{Dimensional form}$$

$$\boxed{\frac{d\theta^*}{dx} + \left(\frac{2H^{**}}{H^*} + 3 - M_e^2\right)\frac{\theta^*}{u_e}\frac{du_e}{dx} = 2C_D} \qquad \text{Dimensionless form}$$

Definitions

$\theta^* = \int \left(1 - \frac{u^2}{u_e^2}\right)\frac{\rho u}{\rho_e u_e}\, dy$	kinetic energy thickness
$\delta^{**} = \int \left(1 - \frac{\rho}{\rho_e}\right)\frac{u}{u_e}\, dy$	volume flux thickness
$D = \int \tau \frac{\partial u}{\partial y}\, dy$	dissipation integral
$H^* = \frac{\theta^*}{\theta}$	kinetic energy shape parameter
$H^{**} = \frac{\delta^{**}}{\theta}$	density thickness shape parameter
$C_D = \frac{D}{\rho_e u_e^3}$	dissipation coefficient