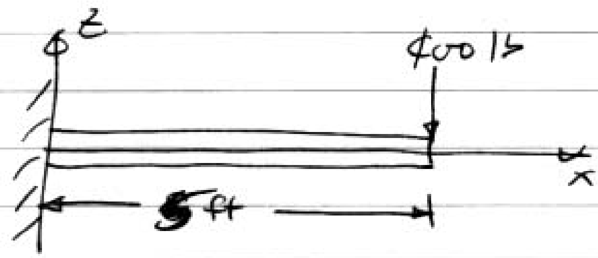
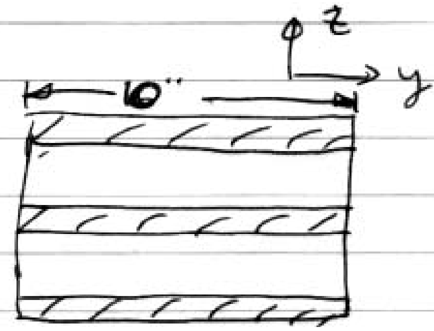
Application Task - (Home Assignment #8)

9.





= 0.1" graphite/epoxy

= 0.4" foam

Material properties:

unidirectional graphite/epoxy: $E_L = 22\,Msi$, $E_T = 3.5\,Msi$, $G_{LT} = 0.8\,Msi$, $\nu_{LT} = 0.28$

Styrofoam: $E = 0.50\,Msi$; $\nu = 0.35$

$$\Rightarrow G = \frac{E}{2(1+\nu)} = 0.19\ Msi$$

a) Find maximum stress, σ_{xx}, and its location

As for the previous problems, we need to find the centroid of the cross-section. In this case, the centroid can be seen to be located at the center of the cross-section due to symmetry. The next step is to find the moments of inertia I_y^*, I_z^*, and I_{yz}^*. Since two materials are used, we need to use modulus-weighted properties. The symmetry of the cross-section also shows that the cross product of inertia, I_{yz}^*, is zero. We can also see that we will not need I_z^* since when $I_{yz}^* = 0$, the beam deflection along the line aligned with the line of load application. Here this is parallel to the z-axis, so we only need I_y^*. We know:

$$I_y^* = \iint z^2 \frac{E}{E_1} \, dA \qquad (1)$$

(use E_L of graphite/epoxy as E_1)

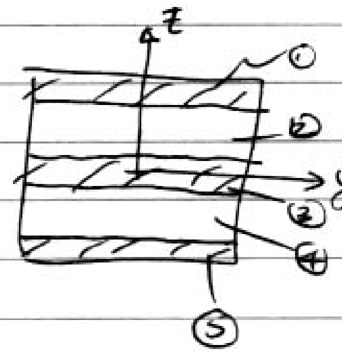
We can find stress using (from Unit #14 p.21):

$$\sigma_{xx} = -\frac{M_y z}{I_y^*} E^* \qquad (2) \quad \left(\text{where: } E^* = \frac{E}{E_1}\right)$$

<u>Moment of inertia, I_y^*</u>

As for previous problems, equation (1) can be simplified to:

$$I_y^* = \sum \left(I_{y_i}^* + A_i z_i^2 \right) \qquad (3)$$



$$I_{y_1}^* = \frac{22\,Msi}{22\,Msi} \left[\frac{1}{12} (10'')(0.1'')^3 + (10'')(0.1'')(0.5'')^2 \right]$$

$$= 0.2508 \text{ in}^4$$

$$\therefore \; I_{y_2}^* = \frac{0.5\,Msi}{22\,Msi} \left[\frac{1}{12} (10'')(0.4'')^3 + (10'')(0.4'')(0.5''-0.05'')^2 \right]$$

$$= 0.0069 \text{ in}^4$$

$$I^*_{y(3)} = \frac{22\,\text{Msi}}{22\,\text{Msi}}\left[\frac{1}{12}(10'')(0.1'')^3 + (10'')(0.1'')(0'')^2\right]$$

$$= 0.0008 \text{ in}^4$$

from symmetry: $I^*_{y(5)} = I^*_{y(1)} = 0.2508 \text{ in}^4$

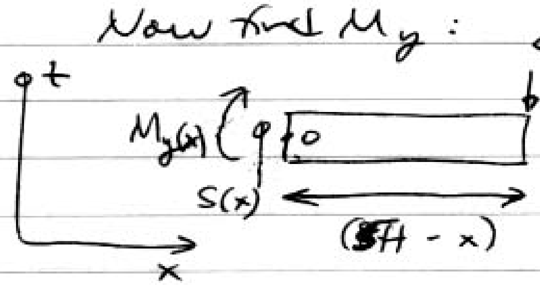
$$I^*_{y(4)} = I^*_{y(2)} = 0.0069 \text{ in}^4$$

Using these results in equation (3):

$$I^*_y = 2(0.2508 \text{ in}^4) + 2(0.0069 \text{ in}^4) + 0.0008 \text{ in}^4$$

$$\Rightarrow I^*_y = 0.5162 \text{ in}^4$$

Now find M_y:



$$\Sigma M_o = 0 \Rightarrow \circlearrowleft +$$

$$-M_y - (5ft - x)400\,lb = 0$$

$$\Rightarrow M_y = -2000 \text{ ft}\cdot\text{lb} + 400\,\text{lb}(x) \qquad (5)$$

Use these results in equation (2):

$$\sigma_{xx} = -\frac{\{-2000 \text{ ft}\cdot\text{lb} + (400\,\text{lb})x\}}{0.5162 \text{ in}^4} z E^* \qquad (6)$$

One can see that M_y is maximum at $x = 0$, so the maximum stress occurs at $x = 0$:

$$\sigma_{xx} = \frac{2000 \text{ ft}\cdot\text{lb}}{0.5162 \text{ in}^4} z E^* \qquad (7)$$

$E^* = \frac{E}{E_1} = \frac{E}{E_{Lr}}$ so the maximum stress occurs at $z = \pm 0.55''$ where $E = E_L \Rightarrow E^* = 1$. Thus, $(\sigma_{xx})_{max}$ occurs at the top and bottom of the cross-section at the root of the beam:

$$(\sigma_{xx})_{max} = \frac{(2000 \text{ ft}\cdot\text{lb})\left(\frac{12 \text{ in}}{\text{ft}}\right)(\pm 0.55'')}{0.5162 \text{ in}^4} \quad (1)$$

$$\Rightarrow \boxed{(\sigma_{xx})_{max} = 25.6 \text{ ksi}} \quad \boxed{@\ x = 0,\ z = \pm 0.55''}$$

Note: σ_{xx} is negative @ $z = -0.55''$ (bottom)
is positive @ $z = +0.55''$ (top)

6) Find the through-thickness distribution of the shear stress and its maximum value and locate

The expression for shear stress, σ_{xz}, for a "general" symmetric cross-section with two or more materials can be derived in a manner similar to that for "simple" (i.e. with only one material) symmetric cross-sections as derived in Unit #13 (pp. 15+16)

The equilibrium equation in the x-direction for a symmetric cross-section beam can be written as:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \quad \text{(same as equation 13-5 in Unit #13)}$$

$$\Rightarrow \frac{\partial \sigma_{xz}}{\partial z} = -\frac{\partial \sigma_{xx}}{\partial x}$$

Multiply this by the width of the beam, b and integrate from z to $\frac{h}{2}$ (outer edge of beam) to get:

$$\int_z^{h/2} b \frac{\partial \sigma_{xz}}{\partial z} dz = -\int_z^{h/2} b \frac{\partial \sigma_{xx}}{\partial x} dz$$

From equation (2) we know that $\sigma_{xx} = -\frac{M_y z}{I_y^*} E^*$.
Use this to get:

$$\int_z^{h/2} b \frac{\partial \sigma_{xz}}{\partial z} dz = -\int_z^{h/2} b \frac{\partial}{\partial x}\left(-\frac{M_y z}{I_y^*} E^*\right) dz$$

integrating yields:

$$b\left[\sigma_{xz}\left(\frac{h}{2}\right) - \sigma_{xz}(z)\right] = -\int_z^{h/2} b E^* \frac{z}{I_y^*} \frac{\partial}{\partial x}(-M_y) dz$$

and $\sigma_{xz} = 0$ at surface (no stress) $\Rightarrow \sigma_{xz}(h/2) = 0$

From equation (5), M_y is only a function of x. And we know that: $\frac{\partial}{\partial x}(-M_y) = -\frac{dM_y}{dx} = -S$

Thus, we can rewrite the above equation as:

$$b\left[-\sigma_{xz}(z)\right] = \int_z^{h/2} b \frac{E^* z}{I_y^*} S \, dz$$

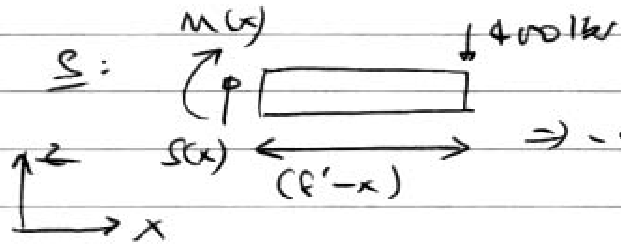
Note that I_y^* is a constant, S is a constant here (although it can be a function of x in general for symmetric cross-sections) and E^* is a function of z. Thus, the equation for shear stress becomes:

$$\sigma_{xz} = -\frac{S Q^*}{I_y^* b} \qquad (8)$$

where: $Q^* = \int_z^{h/2} E^* z b \, dz$

Use equation (8) to calculate the shear stress. First find the shear force:

S:



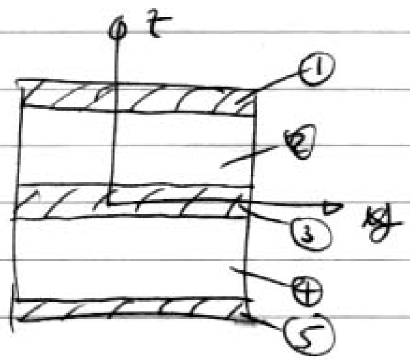
$\Sigma F_z = 0 \; \uparrow +$

$\Rightarrow -400\,\text{lbs} + S(x) = 0$

$\Rightarrow S(x) = 400\,\text{lbs} \quad (9)$

Find Q^*:

Do this in each layer, due to changing E^*



① : $0.55'' \geq z \geq 0.45''$

$$Q^* = \int_z^{0.55''} \left(\frac{22\,\text{Msi}}{22\,\text{Msi}}\right) z (10'') \, dz$$

$$= \frac{z^2 (10'')}{2} \Big]_z^{0.55''}$$

$$= 5\,\text{in} \left(0.325\,\text{in}^2 - z^2\right)$$

② : $0.45'' \geq z \geq 0.05''$

$$Q^* = \int_{0.45''}^{0.55''} \left(\frac{22\,\text{Msi}}{22\,\text{Msi}}\right) z (10'') \, dz + \int_z^{0.45''} \left(\frac{0.5\,\text{Msi}}{22\,\text{Msi}}\right) z (10'') \, dz$$

$$\Rightarrow Q^* = 0.6125\ in^3 + (5\ in)(0.023)(0.2025\ in^2 - z^2)$$

3: $0.05'' \geq z \geq -0.05''$

$$Q^* = \int_{0.45''}^{0.55''} \left(\frac{22\,Msi}{22\,Msi}\right) z(10'')\,dz + \int_{0.05''}^{0.45''} \left(\frac{0.5\,Msi}{22\,Msi}\right) z(10'')\,dz + \int_{z}^{0.05''} \left(\frac{22\,Msi}{22\,Msi}\right) z(10'')\,dz$$

$$= 0.6125\ in^3 + 0.023\ in^3 + 5\,in\,(0.0025\ in^2 - z^2)$$

4: $-0.05'' \geq z \geq -0.45''$

$$Q^* = \int_{0.45''}^{0.55''} \left(\frac{22\,Msi}{22\,Msi}\right) z(10'')\,dz + \int_{0.05''}^{0.45''} \left(\frac{0.5\,Msi}{22\,Msi}\right) z(10'')\,dz + \int_{-0.05''}^{0.05''} \left(\frac{22\,Msi}{22\,Msi}\right) z(10'')\,dz + \int_{z}^{-0.05''} \left(\frac{0.5\,Msi}{22\,Msi}\right) z(10'')\,dz$$

$$= 0.6125\ in^3 + 0.023\ in^3 + 0 + (5\ in)(0.023)(0.0025\ in^2 - z^2)$$

5: $-0.45'' \geq z \geq 0.55''$

$$Q^* = \int_{0.45''}^{0.55''} \left(\frac{22\,Msi}{22\,Msi}\right) z(10'')\,dz + \int_{0.05''}^{0.45''} \left(\frac{0.5\,Msi}{22\,Msi}\right) z(10'')\,dz + \int_{-0.05''}^{0.05''} \left(\frac{22\,Msi}{22\,Msi}\right) z(10'')\,dz + \int_{-0.45''}^{-0.05''} \left(\frac{0.5\,Msi}{22\,Msi}\right) z(10'')\,dz + \int_{z}^{-0.45''} \left(\frac{22\,Msi}{22\,Msi}\right) z(10'')\,dz$$

$$= 0.6125\ in^3 + 0.023\ in^3 + 0 + (-0.023\ in^3) + (5\ in)(0.2025\ in^2 - z^2)$$

Looking at the expression for shear stress in equation (8), we can see that the shear stress is a maximum when Q^* is a maximum. We can see that Q^* is parabolic in each layer and $= 0$ at the top and bottom surface of the cross-section, and is symmetric about the y-axis. Thus, the maximum occurs at $z = 0$

$$\Rightarrow (Q^*)_{max} = 0.648 \text{ in}^3 \quad @ \; z = 0$$

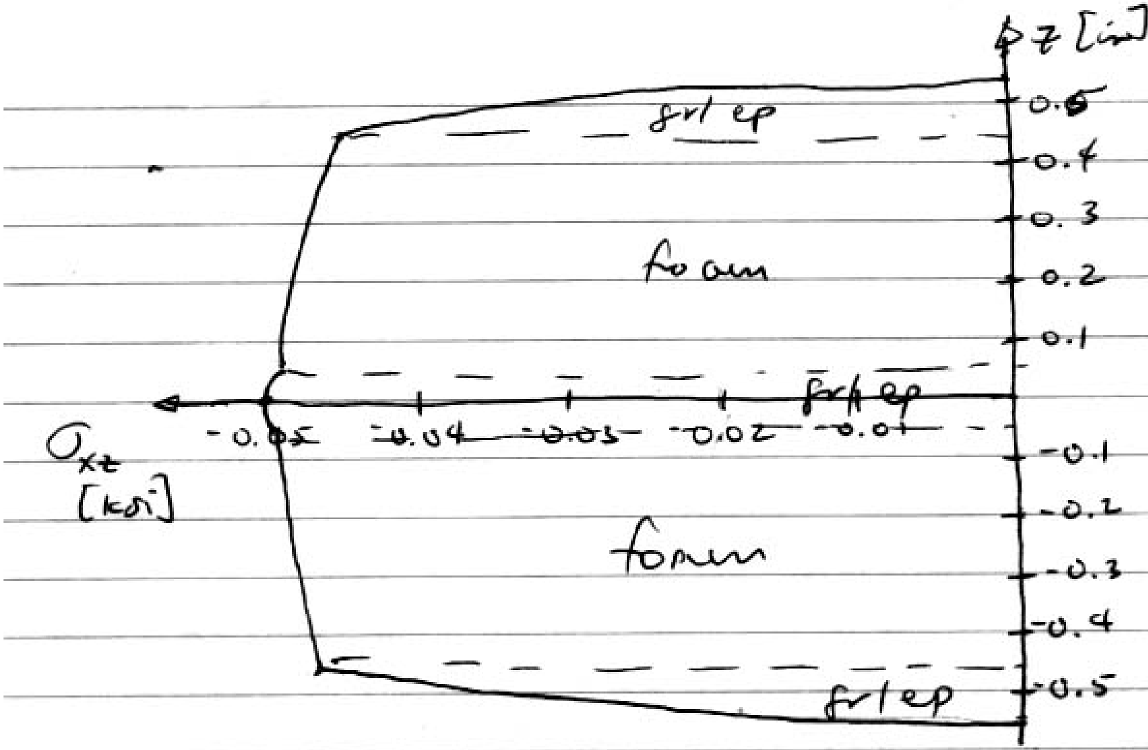
and use: $$\sigma_{xz} = -\frac{S Q^*}{I_y^* b}$$

$$\Rightarrow (\sigma_{xz})_{max} = -\frac{(400 \text{ lb})(0.648 \text{ in}^3)}{(0.5162 \text{ in}^4)(10 \text{ in})}$$

$$\Rightarrow \boxed{(\sigma_{xz})_{max} = -0.050 \text{ ksi}}$$

$\boxed{@ \; z = 0 \text{ everywhere along } x}$

The plot of σ_{xz} through-the-thickness follows. It has the same characteristics as Q^* as it is parabolic in nature, equal to zero at the top and bottom surfaces, symmetric about the y-axis, and continuous with a change in ~~[illegible]~~ "slopes" with each layer.



c) Find the maximum deflection

The deflection has two components: bending and shear.

Consider bending first:

$$\frac{d^2 w_B}{dx^2} = \frac{M_y}{E_1 I_y^*} \qquad \text{(from Unit \#14, p. 19)}$$

$$\Rightarrow \frac{d^2 w_B}{dx^2} = \frac{[-2400 \text{ in·lb} + (400 \text{ lb})x]}{E_1 I_y^*}$$

$$\Rightarrow \frac{dw_B}{dx} = \frac{1}{E_1 I_y^*}\left[(-2400 \text{ in·lb})x + (200 \text{ lb})x^2 + C_1\right]$$

$$\Rightarrow w_B = \frac{1}{E_1 I_y^*}\left[(-1200 \text{ in·lb})x^2 + \left(\frac{200}{3}\text{ lb}\right)x^3 + C_1 x + C_2\right]$$

The boundary conditions are a clamped condition at $x = 0$ $\Rightarrow$

$$\frac{dw}{dx}(0) = 0 \quad \Rightarrow \quad C_1 = 0$$

$$w(0) = 0 \quad \Rightarrow \quad C_2 = 0$$

Thus, the deflection due to bending is:

$$w_B(x) = \frac{\left[(-12000\text{ in}\cdot\text{lb})\,x^2 + \left(\frac{200\text{ lb}}{3}\right)x^3\right]}{\left(22\times10^6\ \frac{\text{lb}}{\text{in}^2}\right)(0.062\text{ in}^4)}$$

The maximum deflection due to bending occurs at the tip of the beam (@ $x = 8' = 96''$)

$$(w_B)_{max} = \frac{1}{1\ \times10^6(\text{lb}\cdot\text{in}^2)}\Big[(-12000\text{ in}\cdot\text{lb})(60\text{ in})^2 + \left(\frac{200\text{ lb}}{3}\right)(60\text{ in})^3\Big]$$

$$\Rightarrow \boxed{(w_B)_{max} = -3.79\text{ in}} \quad @\ x = 5\text{ ft}$$

Consider the shear deflection. The pertinent equation is:

$$\frac{dw_s}{dx} = -\frac{S}{GA_e}$$

(from Unit #14, p. 26
NOTE: This equation derived for "average" $\frac{dw_s}{dx}$)

This equation does not indicate how we should consider varying the shear modulus across the thickness of the beam. However note that GA_e is the "shear stiffness" of the beam (this concept is similar to the bending stiffness of the beam, EI from $\frac{d^2 w_b}{dx^2} = \frac{M}{EI}$). If we compare the relative shear stiffnesses of the beam, we find:

factor for effective area for rectangular

(GA_e) of graphite epoxy $= (0.8 \text{ Msi})(0.83)(10'')(0.1'')(3)$
$= 2.0 \times 10^6$ lbs

(GA_e) of foam $= (0.19 \text{ Msi})(0.83)(10'')(0.8'')(3)$
$= 1.26 \times 10^6$ lbs

Let's use the smaller value to be conservative since foam is less resistant to shear deformation and there is more of it

This gives: $\frac{dw_s}{dx} = -\frac{S}{GA_e}$

$$= -\frac{(400 \text{ lb})}{(1.26 \times 10^6 \text{ lbs})}$$

$$\Rightarrow w_s(x) = -3.17 \times 10^{-4}\, x + C_1$$

With a clamped root, the pertinent boundary condition is $w(0) = 0 \Rightarrow C_1 = 0$

$\therefore$

So: $w_s(x) = -3.17 \times 10^{-4}\, x$

The maximum occurs at $x = 60''$ (5 feet)

$$\boxed{w_{S(max)} = -0.019 \text{ in}}$$

$\boxed{\text{at } x = 60''}$

This is over 2 orders of magnitude smaller than the bending deflection even using the conservative value for GA_e. Thus, we approximate (a good approximation to 1%) the total deflection as the bending deflection:

$$\boxed{(w)_{max} \approx (w_B)_{max} = -3.79 \text{ in}}$$