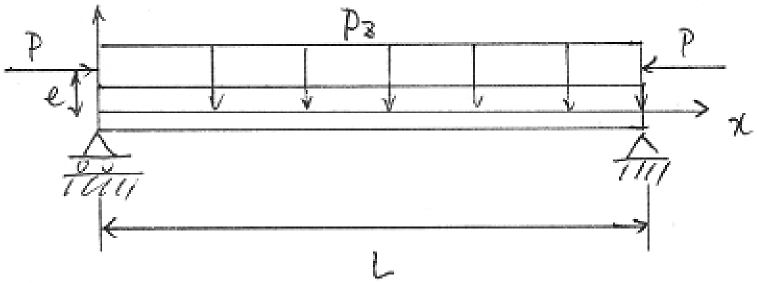
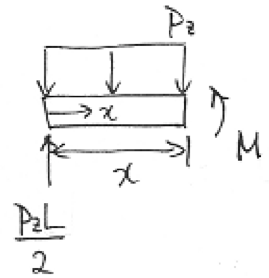
Application Tasks

5.



cross-sectional property : EI

a) Let's consider the primary bending moment due to p_z first.

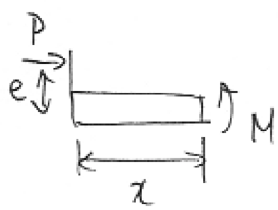


$\Sigma M = 0$: $M = -p_z x \frac{1}{2} x + \frac{p_z L}{2} x$
@ $x=0$

$\Rightarrow \boxed{M = -\frac{1}{2} p_z x^2 + \frac{1}{2} p_z L x}$

primary bending moment from p_z

Next, let's consider the primary bending moment due to the axial load



$\Sigma M = 0$: $M = Pe$
@ $x=0$

$\Rightarrow \boxed{M = Pe}$

primary bending moment from P

b) The governing equation can be obtained from

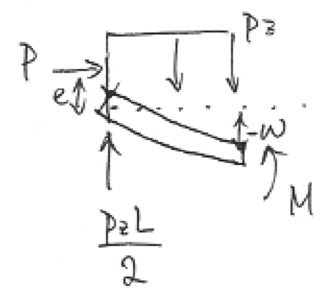
$$EI \frac{d^2 w}{dx^2} + Pw = M_{primary} \qquad ③$$

* see unit 17, p.12

Thus, from part a), the governing equation is

$$\boxed{EI\frac{d^2w}{dx^2} + Pw = Pe + \frac{1}{2}p_z(Lx - x^2)} \qquad \text{④}$$

Note that the govering equation can also be obtained through the procedure shown in unit 17, p.11.



$$\Sigma M = 0 \; : \; \underbrace{M_{primary}}_{\text{from eqs. ① \& ②}} - \underbrace{Pw}_{\text{secondary bending moment}} = \underbrace{EI\frac{d^2w}{dx^2}}_{M}$$

$$\Rightarrow EI\frac{d^2w}{dx^2} + Pw = Pe + \frac{1}{2}p_z(Lx - x^2)$$

c) Let's consider the homogeneous part first. The solution is of the form

$$w_h(x) = A\sin\lambda x + B\cos\lambda x \qquad \text{⑤}$$

w_h: homogeneous

$$\lambda = \sqrt{\frac{P}{EI}}$$

The particular solution is a polynomial of the form

$$w_p(x) = Cx^2 + Dx + F \qquad \text{⑥}$$

Plugging equation ⑥ into the governing equation ④, we get the cons[tants] C, D and F.

$$EI(2C) + P(Cx^2 + Dx + F) = Pe + \frac{1}{2}p_zLx - \frac{1}{2}p_zx^2$$

$$\Rightarrow \; PCx^2 + PDx + (PF + 2EIC) = -\frac{1}{2}p_z x^2 + \frac{1}{2}p_z L x + Pe$$

Matching coefficients, we get,

$$x^2: \quad PC = -\frac{1}{2}p_z \quad \Rightarrow \quad C = -\frac{1}{2}\frac{p_z}{P} \quad \text{——— ⑦}$$

$$x: \quad PD = \frac{1}{2}p_z L \quad \Rightarrow \quad D = \frac{1}{2}\frac{p_z L}{P} \quad \text{——— ⑧}$$

from eq.⑦

$$x^0: \quad PF + 2EIC = Pe \quad \Rightarrow \quad F = e - \cancel{2}EI\frac{1}{P}\left(-\frac{1}{\cancel{2}}\frac{p_z}{P}\right)$$

$$\Rightarrow \quad F = e + \frac{p_z EI}{P^2} \quad \text{——— ⑨}$$

Plugging these constants back into the particular solution in equatio[n] ⑥ and adding the homogeneous part in equation ⑤, we get the total deflection.

$$w(x) = A\sin\lambda x + B\cos\lambda x - \frac{1}{2}\frac{p_z}{P}x^2 + \frac{1}{2}\frac{p_z L}{P}x + e + \frac{p_z EI}{P^2} \quad \text{——(}$$

The boundary conditions @ $x=0$ and $x=L$ are needed to find A and B. Since the beam-column is simply-supported at both ends, the boundary conditions are that there is no deflection, i.e., $w=0$

$$w(0)=0: \quad w(0) = B + e + \frac{p_z EI}{P} = 0$$

$$\Rightarrow \quad B = -e - \frac{p_z EI}{\ldots} \quad \text{——— ⑪}$$

$w(L) = 0$: $\quad w(L) = A\sin\lambda L + \left(-e - \frac{P_z EI}{P^2}\right)\cos\lambda L$

$$- \cancel{\frac{1}{2}\frac{P_z}{P}L^2} + \cancel{\frac{1}{2}\frac{P_z}{P}L^2} + e + \frac{P_z EI}{P^2} = 0$$

$$\Rightarrow \quad A = \left(e + \frac{P_z EI}{P^2}\right)(\cos\lambda L - 1)\frac{1}{\sin\lambda L} \quad \text{———} \quad (12)$$

Therefore, the final solution is

$$\boxed{w(x) = \left(e + \frac{P_z EI}{P^2}\right)\left(\frac{\cos\lambda L - 1}{\sin\lambda L}\sin\lambda x - \cos\lambda x + 1\right) - \frac{1}{2}\frac{P_z}{P}x(x-L)} \text{ —}$$

d) The mid-span deflection is

$$w\left(\frac{L}{2}\right) = \left(e + \frac{P_z EI}{P^2}\right)\left(\frac{\cos\lambda L - 1}{\sin\lambda L}\sin\frac{\lambda L}{2} - \cos\frac{\lambda L}{2} + 1\right) + \frac{1}{8}\frac{P_z}{P}L^2$$

$$\Rightarrow \quad w\left(\frac{L}{2}\right) \triangleq w_c = \left(e + \frac{P_z EI}{P^2}\right)\left(\frac{\cos\lambda L - 1}{\sin\lambda L}\sin\frac{\lambda L}{2} - \cos\frac{\lambda L}{2} + 1\right) + \frac{1}{8}\frac{P_z}{P}L^2$$

Let's normalized the mid-span deflection with L by dividing through by L.

$$\frac{w_c}{L} = \left(\frac{e}{L} + \frac{P_z EI}{P^2 L}\right)\left(\frac{\cos\lambda L - 1}{\sin\lambda L}\sin\frac{\lambda L}{2} - \cos\frac{\lambda L}{2} + 1\right) + \frac{1}{8}\frac{P_z}{P}L \quad \text{—} \quad (14)$$

Next, we'll normalized the load with P_{cr} for the simply-supported case which is

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad \text{———} \quad (15)$$

Consider the terms that contain the load P.

$$\frac{p_z EI}{P^2 L} = \frac{p_z \cancel{EI}}{P^2 \cancel{L}} \frac{P_{cr}^2}{P_{cr}\left(\frac{\pi^2 \cancel{EI}}{L^{\cancel{2}}}\right)} = \left(\frac{P_{cr}}{P}\right)^2 \frac{p_z L}{P_{cr}\pi^2}$$

$$\lambda L = \sqrt{\frac{P}{EI}}\, L = \sqrt{\frac{P}{\cancel{EI}}}\, \cancel{L} \sqrt{\frac{\pi^2 \cancel{EI}}{L^{\cancel{2}} P_{cr}}} = \pi \sqrt{\frac{P}{P_{cr}}}$$

$$\frac{p_z L}{P} = \frac{p_z L}{P} \frac{P_{cr}}{P_{cr}} = \left(\frac{P_{cr}}{P}\right)\left(\frac{p_z L}{P_{cr}}\right)$$

Substituting these equations back into equation (14), we get

$$\frac{w_c}{L} = \left(\frac{e}{L} + \left(\frac{P_{cr}}{P}\right)^2 \frac{p_z L}{P_{cr}\pi^2}\right)\left(\frac{\cos \pi\sqrt{\frac{P}{P_{cr}}} - 1}{\sin \pi\sqrt{\frac{P}{P_{cr}}}} \sin \frac{1}{2}\pi\sqrt{\frac{P}{P_{cr}}} - \cos\frac{1}{2}\pi\sqrt{\frac{P}{P_{cr}}} + 1\right)$$

$$+ \frac{1}{8}\left(\frac{P_{cr}}{P}\right)\left(\frac{p_z L}{P_{cr}}\right) \qquad \text{——} \ (1$$

If we define new variables with

$$\tilde{w} = \frac{w_c}{L} \qquad \tilde{P} = P \qquad \tilde{e} = \frac{e}{L} \qquad \tilde{p}_z = \frac{p_z L}{P_{cr}},$$

equation (15) becomes

$$\tilde{w} = \left(\tilde{e} + \frac{1}{\tilde{P}^2}\frac{\tilde{p}_z}{\pi^2}\right)\left(\frac{\cos \pi\sqrt{\tilde{P}} - 1}{\sin \pi\sqrt{\tilde{P}}} \sin \frac{\pi}{2}\sqrt{\tilde{P}} - \cos\frac{\pi}{2}\sqrt{\tilde{P}} + 1\right)$$

$$+ \frac{1}{8}\frac{\tilde{p}_z}{\tilde{P}} \qquad \text{——} \ (11)$$

The following cases are considered for the parametric study.

I. $\hat{e} = \frac{e}{L} = 0.01\ ,\ 0.05\ ,\ 0.1$ for $\hat{p}_z = 0$

II. $\hat{p}_z = \frac{p_z L}{P_{cr}} = 0.1,\ 0.2,\ 0.5,\ 1$ for $\hat{e} = 0$

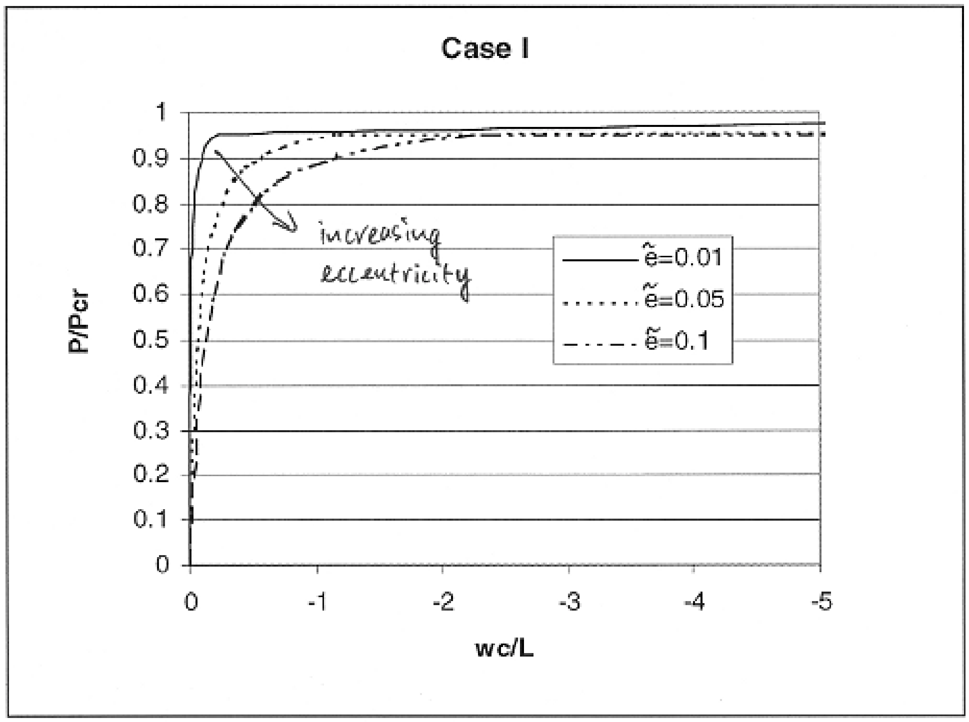
III. $\hat{e} = 0.1\ ,\ \hat{p}_z = 0.1,\ 0.2,\ 0.5,\ 1$

IV $\hat{p}_z = 1\ ,\ \hat{e} = 0.01,\ 0.05,\ 0.1$

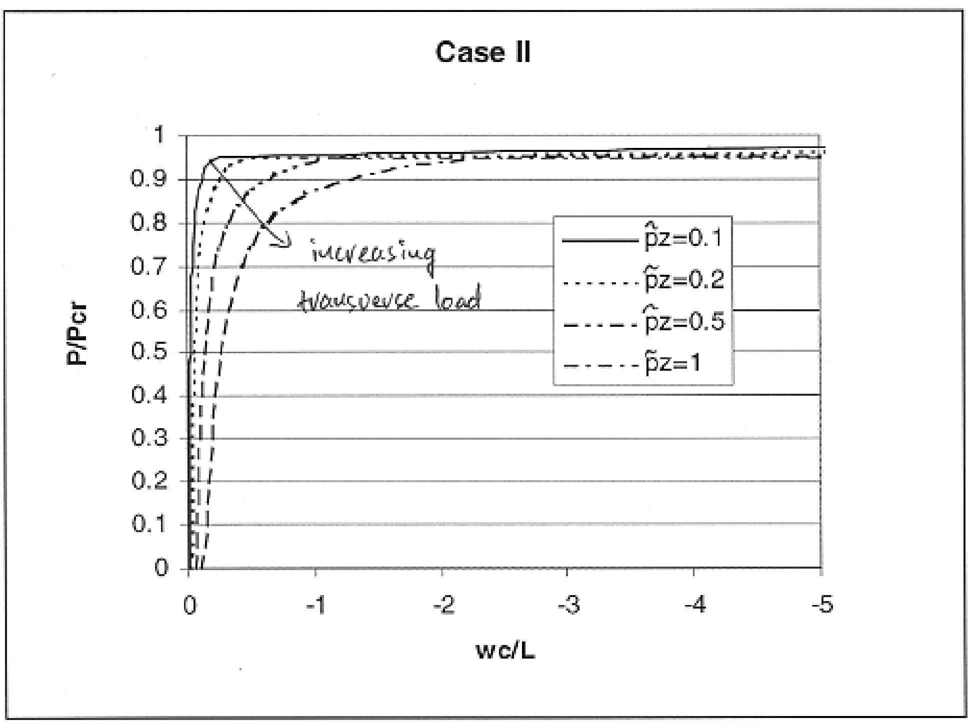
$\Big\}$ cross-cases.

The plots are shown in the following pages. From the plots we find

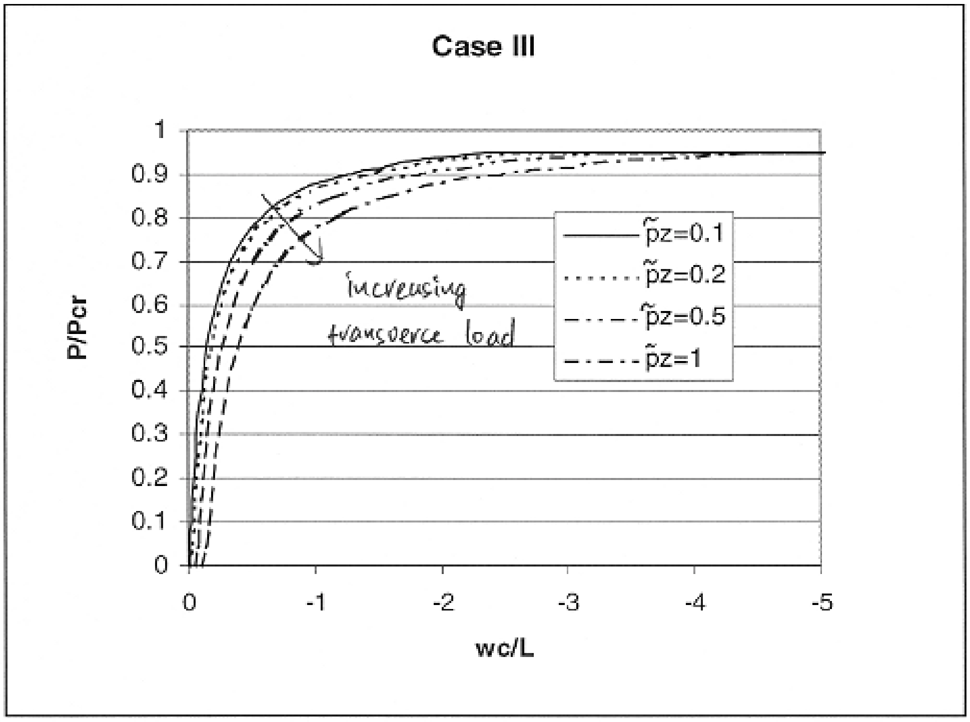
- General trend is that increasing $\hat{e}$. or $\hat{p}_z$ will increase the deflection for any given load
- This makes sense because both the transverse load p_z and the eccentricity, e, produce moments in the same direction, magnifying the pre-buckling deflection.
- Most importantly, the eccentricity varies how spread out the curves are, while the transverse load affects the amount of initial deflection directly.



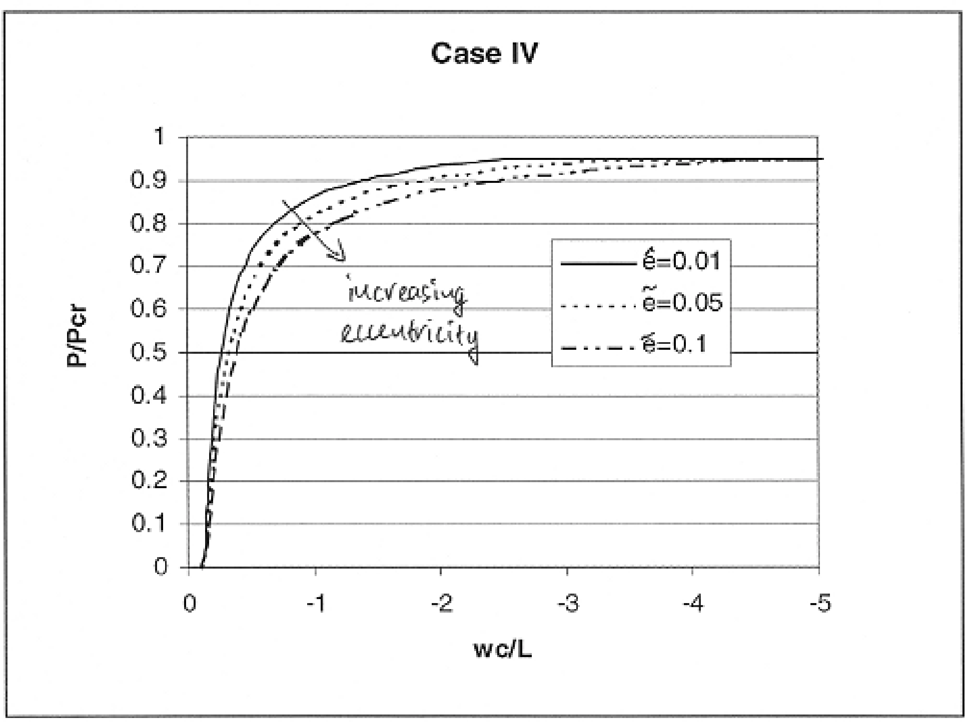
$\tilde{p}_z = 0$



$\hat{e} = 0$



$\tilde{e} = 0.1$



$\tilde{p}_z = 1$