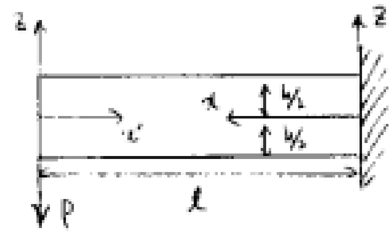
Application Task

6. a)



To use the given Airy stress function sheet, lets define another coordinate system, $x'-z$ coordinate, where

$$x' = l - x \qquad (1)$$

The boundary conditions are :

i) No applied tractions on the top and bottom

$$\Rightarrow \quad \sigma_{zz} = \sigma_{xz} = 0 \quad @ \quad z = \pm\frac{h}{2}$$

ii) Forces at tip (applied)

$$\Rightarrow \quad \sigma_{xx} = 0 , \quad \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} b\,dz = P \quad @ \quad x' = 0 \ (\text{or } x = l)$$

iii) Reactions at wall

$$\Rightarrow \quad \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} b\,dz = P , \quad \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}\, z\, b\,dz = Pl \quad @ \quad x' = l \ (\text{or } x = 0)$$

* Note : In the Airy stress function sheet, ϕ's are given for x and y. Although our problem is formulated in the $x-z$ coordinate, it is still a plane stress problem, and we can substitute z for y. The plane stresses in our problem is σ_{xx}, σ_{xz} and σ_{zz}.

Now, let's try to select the stress functions needed to describe the stress field, paying particular attention to the boundary conditions. Based on the boundary conditions and intuition, $\sigma_{zz} = 0$ throughout the beam. Thus, we need to select the stress functions that gives $\sigma_{zz} = 0$. In addition, σ_{xz} should be non-zero based on the boundary condition. Selecting the stress functions that meet these two criteria, we are left with:

$\phi(x', z)$	σ_{xz}	σ_{xx}	σ_{zz}
$\phi_1 = C_{11} x' z$	$-C_{11}$	0	0
$\phi_2 = C_{12} x' z^2$	$-2C_{12} z$	$2C_{12} x'$	0
$\phi_3 = C_{13} x' z^3$	$-3C_{13} z^2$	$6C_{13} x' z$	0

Applying B.C. i), we need

$$\sigma_{xz} = 0 \quad @ \quad z = \pm \frac{h}{2}$$

where σ_{xz} is a combination of the shear stresses from ϕ_1, ϕ_2 and ϕ_3. Since $-2C_{12}z$ (from ϕ_2) is linear in z, ϕ_2 cannot be part of that combination $\Leftarrow$ $-2C_{12}\left(+\frac{h}{2}\right) \neq -2C_{12}\left(-\frac{h}{2}\right)$.

Thus, the only possible combination will be

$$\phi = \phi_1 + \phi_3 = C_{11} x'z + C_{13} x'z^3 \qquad \text{②}$$

and the shear stress is

$$\sigma_{xz} = -C_{11} - 3C_{13} z^2 \qquad \text{③}$$

B.C. i) tells us that (using equation ②)

$$0 = -C_{11} - 3C_{13} \left(\pm\frac{h}{2}\right)^2$$

$$\Rightarrow \quad C_{11} = -\frac{3}{4} C_{13} h^2$$

$$\therefore \quad \phi = C_{13}\left(x'z^3 - \frac{3}{4} h^2 x'z\right) \qquad \text{④}$$

Note that the other B.C. in i), i.e., $\sigma_{zz} = 0$, is already satisfied since we chose the stress functions that satisfied this condition.

B.C. ii) tells us that

$$\sigma_{xx}\Big|_{x'=0} = 6C_{13} x'z\Big|_{x'=0} = 0 \qquad \text{⑤}$$

So, this B.C. is satisfied. The other B.C. in ii) tells us that

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz}\, b\, dz = P \qquad @\ x' = 0$$

$$\Rightarrow \quad b\int_{-\frac{h}{2}}^{\frac{h}{2}} -C_{13}\left(3z^2 - \frac{3}{4}h^2\right)dz = P$$

$$\Rightarrow \quad b\,C_{13}\left[\frac{3}{4}h^2 z - 3\frac{z^3}{3}\right]_{-\frac{h}{2}}^{\frac{h}{2}} = P$$

$$\Rightarrow \quad C_{13}\left(\frac{3}{4}h^3 - \frac{1}{4}h^3\right) = \frac{P}{b}$$

$$\therefore \quad C_{13} = \frac{2P}{bh^3} \qquad \text{------------ } ①$$

$$\therefore \quad \boxed{\begin{gathered}\phi = \frac{2P}{bh^3}\left(x'z^3 - \frac{3}{4}h^2 x' z\right) \\ \text{or} \\ \phi = \frac{2P}{bh^3}\left((l-x)z^3 - \frac{3}{4}h^2(l-x)z\right)\end{gathered}}$$

$$\boxed{\begin{gathered}\sigma_{xz} = \frac{6P}{bh^3}\left(\frac{h^2}{4} - z^2\right) \\ \sigma_{xx} = \frac{12P}{bh^3}x'z \\ \text{or} \\ \sigma_{xx} = \frac{12P}{bh^3}(l-x)z\end{gathered}}$$

* Note that the stresses also satisfy B.C. iii).

b) The stresses from the simple beam theory can be found as follows. Recall from unified that

$$\sigma_{xx} = -\frac{Mz}{I}, \quad \sigma_{zz} = 0, \quad \sigma_{xz} = -\frac{SQ}{Ib} \qquad \text{---- ①} \sim \text{③}$$

for the beam shown below.

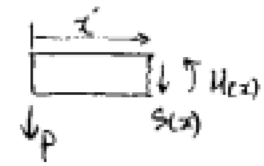
$V = P$

$M = -Pl$

The shear and moment at an arbitrary face at x is:

$$S(x) = -P$$

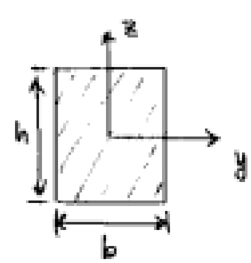
$$M(x) = -Px$$



Look at the cross-section geometry

$$I = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 b\,dz = \frac{bh^3}{12}$$

$$Q = \int_{z}^{h/2} z b\,dz = \frac{b}{2}\left(\frac{h^2}{4} - z^2\right)$$



Substituting the appropriate values into equations ① through ③, we get

$$\sigma_{xx} = -\frac{M_z}{I} = -\frac{(-Px')z}{bh^3/12} = \frac{12P}{bh^3}x'z \quad \longleftarrow \text{same as in a)}$$

$$\sigma_{xz} = -\frac{(-P)\frac{b}{2}\left(\frac{h^2}{4}-z^2\right)}{(bh^3/12)\,b} = \frac{6P}{bh^3}\left(\frac{h^2}{4}-z^2\right) \quad \longleftarrow \text{same as in a)}$$

$$\sigma_{zz} = 0 \quad \longleftarrow \text{same as in a)}$$

Therefore, the stresses obtained using the simple beam theory is identical to those obtained using the Airy stress functions.

c) There are two locations where we should apply St. Venant's Principle in this problem, and both involve the boundaries.

1. At $x'=0$, we have the shear load P, which is equally distributed across the height, whereas our stress σ_{xz} is a parabolic distribution. Thus, we ignored the specifics of the load distribution and applied St. Venant's principle.

2. At $x'=l$ (the wall), we have a linear distribution of σ_{xx}. However, we don't know the specifics of the load distribution at the wall. We represent the reaction by the equipollent force system with a moment, a vertical load, and a zero horizontal

load. Thus, once again we satisfy the load condition only in an average sense and invoke St. Venant's principle.