Application Tasks

5. The 2-D stress-strain equation for an orthotropic material can be written as:

$$\sigma_{\alpha\beta} = E^*_{\alpha\beta\sigma\gamma}\,\epsilon_{\sigma\gamma} \Rightarrow \quad \sigma_{11} = E^*_{1111}\,\epsilon_{11} + E^*_{1122}\,\epsilon_{22} \quad (1)$$

$$\sigma_{22} = E^*_{1122}\,\epsilon_{11} + E^*_{2222}\,\epsilon_{22} \quad (2)$$

$$\sigma_{12} = 2\,E^*_{1212}\,\epsilon_{12} \quad (3)$$

Next, write the stress-strain equations in terms of the strains (compliances! See Unit 5, p. 17)

$$\epsilon_{11} = \frac{1}{E_L}\sigma_{11} - \frac{\nu_{LT}}{E_L}\sigma_{22} - \frac{\nu_{L3}}{E_L}\sigma_{33} \quad (4)$$

$$\epsilon_{22} = -\frac{\nu_{TL}}{E_L}\sigma_{11} + \frac{1}{E_T}\sigma_{22} - \frac{\nu_{T3}}{E_T}\sigma_{33} \quad (5)$$

$$\epsilon_{33} = -\frac{\nu_{L3}}{E_L}\sigma_{11} - \frac{\nu_{T3}}{E_T}\sigma_{22} + \frac{1}{E_3}\sigma_{33} \quad (6)$$

$$\epsilon_{23} = \frac{1}{2G_{T3}}\sigma_{23} \quad (7)$$

$$\epsilon_{13} = \frac{1}{2G_{L3}}\sigma_{13} \quad (8)$$

$$\epsilon_{12} = \frac{1}{2G_{LT}}\sigma_{12} \quad (9)$$

* Note that the subscripts L an T correspond to

1 and 2, respectively.

Now substitute the plane stress condition of

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \qquad (10)$$

into equations (4) - (9) to get:

$$(4) \rightarrow \epsilon_{11} = \frac{1}{E_L}\sigma_{11} - \frac{\nu_{LT}}{E_L}\sigma_{22} \qquad (11)$$

$$(5) \rightarrow \epsilon_{22} = \frac{-\nu_{TL}}{E_T}\sigma_{11} + \frac{1}{E_T}\sigma_{22} \qquad (12)$$

$$(6) \rightarrow \epsilon_{33} = \frac{-\nu_{L3}}{E_L}\sigma_{11} - \frac{\nu_{T3}}{E_T}\sigma_{22} \qquad (13)$$

$$(7) \rightarrow \epsilon_{23} = 0 \qquad (14)$$

$$(8) \rightarrow \epsilon_{13} = 0 \qquad (15)$$

$$(9) \rightarrow \epsilon_{12} = \frac{1}{2G_{LT}}\sigma_{12} \qquad (16)$$

The out-of-plane deformation is expressed in equation (13) but is not of primary interest in this problem. Neither are the out-of-plane shear strains which are zero as found in equations (14) and (15).

We can now relate the 2-D stiffness tensor, $E^{*}_{\alpha\beta\sigma\delta}$, to the elastic constants by comparing equations

(1) - (3) with equations (11), (12) and (16). Use matrix form to make the comparison straightforward.

For equations (1) - (3):

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} E^*_{1111} & E^*_{1122} & 0 \\ E^*_{1122} & E^*_{2222} & 0 \\ 0 & 0 & E^*_{1212} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} \qquad (17)$$

Similarly for (11), (12) and (16):

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} 1/E_L & -\nu_{LT}/E_L & 0 \\ -\nu_{TL}/E_T & 1/E_T & 0 \\ 0 & 0 & 1/2G_{LT} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \qquad (18)$$

(2-D compliance form: $\underline{\epsilon} = \underline{S}\, \underline{\sigma}$)

In order to compare equation (17) with equation (18) one of the equations needs to be inverted.

Choosing equation (17):

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} \dfrac{E^*_{2222}}{E^*_{1111}E^*_{2222} - (E^*_{1122})^2} & \dfrac{-E^*_{1122}}{E^*_{1111}E^*_{2222} - (E^*_{1122})^2} & 0 \\ \dfrac{-E^*_{1122}}{E^*_{1111}E^*_{2222} - (E^*_{1122})^2} & \dfrac{E^*_{1111}}{E^*_{1111}E^*_{2222} - (E^*_{1122})^2} & 0 \\ 0 & 0 & \dfrac{1}{2E^*_{1212}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

$\hookrightarrow$ (19)

Now compare the elements of equations (18) and (19):

- $$\frac{1}{E_L} = \frac{E^*_{2222}}{E^*_{1111}E^*_{2222} - (E^*_{1122})^2}$$

$$\Rightarrow \quad E_L = E^*_{1111} - \frac{(E^*_{1122})^2}{E^*_{2222}} \qquad (20a)$$

- $$-\frac{\nu_{LT}}{E_L} = -\frac{E^*_{1122}}{E^*_{1111}E^*_{2222} - (E^*_{1122})^2}$$

$$\Rightarrow \quad \nu_{LT} = \frac{E^*_{1122}}{E^*_{1111}E^*_{2222} - (E^*_{1122})^2} E_L$$

Using the result above of equation (20a):

$$\nu_{LT} = \frac{E^*_{1122}}{E^*_{1111}E^*_{2222} - (E^*_{1122})^2} \frac{E^*_{1111}E^*_{2222} - (E^*_{1122})^2}{E^*_{2222}}$$

$$\Rightarrow \quad \nu_{LT} = \frac{E^*_{1122}}{E^*_{2222}} \qquad (20b)$$

- $$\frac{1}{E_T} = \frac{E^*_{1111}}{E^*_{1111}E^*_{2222} - (E^*_{1122})^2}$$

$$\Rightarrow \quad E_T = E^*_{2222} - \frac{(E^*_{1122})^2}{E^*_{1111}} \qquad (20c)$$

- $$-\frac{\nu_{TL}}{E_T} = -\frac{E^*_{1122}}{E^*_{1111}E^*_{2222} - (E^*_{1122})^2}$$

using the result above of equation (20c):

$$\nu_{TL} = \frac{E^*_{1122}}{E^*_{1111} E^*_{2222} - (E^*_{1122})^2} \cdot \frac{E^*_{1111} E^*_{2222} - (E^*_{1122})^2}{E^*_{1111}}$$

$$\Rightarrow \nu_{TL} = \frac{E^*_{1122}}{E^*_{1111}} \qquad (20\,d)$$

- $$\frac{1}{2G_{LT}} = \frac{1}{2E^*_{1212}}$$

$$\Rightarrow G_{LT} = E^*_{1212} \qquad (20e)$$

Summarizing:

$$\boxed{\begin{aligned} E_L &= E^*_{1111} - \frac{(E^*_{1122})^2}{E^*_{2222}} \\ E_T &= E^*_{2222} - \frac{(E^*_{1122})^2}{E^*_{1111}} \\ \nu_{LT} &= E^*_{1122}/E^*_{2222} \\ \nu_{TL} &= E^*_{1122}/E^*_{1111} \\ G_{LT} &= E^*_{1212} \end{aligned}} \qquad (20\,a\text{-}e)$$

If instead of inverting equation (17) one inverted equation (18) and then did the comparison of equations one would get the relations in inverse: $E^*_{\alpha\beta\gamma\delta}$ in terms of the in-plane engineering constants:

$$\boxed{\begin{aligned} E^*_{1111} &= \frac{E_L}{1-\nu_{LT}\nu_{TL}} \\ E^*_{1122} &= \frac{\nu_{LT}E_T}{1-\nu_{LT}\nu_{TL}} = \frac{\nu_{TL}E_L}{1-\nu_{LT}\nu_{TL}} \\ E^*_{2222} &= \frac{E_T}{1-\nu_{LT}\nu_{TL}} \\ E^*_{1212} &= G_{LT} \end{aligned}} \qquad (21)$$

6. For the isotropic case, the same relation between $E^*_{\alpha\beta\gamma\delta}$ and the elastic constants hold. The differences are that isotropic materials have the same properties in all directions:

$$E_L = E_T = E$$

$$\nu_{LT} = \nu_{TL} = \nu$$

and $E^*_{1111} = E^*_{2222}$ (found in problem #2)

and there is a relationship between E, ν, and G:

$$G = \frac{E}{2(1+\nu)}$$

We also had from Problem #2:

$$E^*_{1212} = \frac{E_{1111} - E_{1122}}{2}$$

with: $E^*_{1111} = E_{1111} - \dfrac{E_{1122}^2}{E_{1111}}$ and $E^*_{1122} = E_{1122} - \dfrac{E_{1122}^2}{E_{1111}}$

So: $E_{1111} = E^*_{1111} + \dfrac{E_{1122}^2}{E_{1111}}$ and $E_{1122} = E^*_{1122} + \dfrac{E_{1122}^2}{E_{1111}}$

using in the expression for E^*_{1212}:

$$E^*_{1212} = \frac{1}{2}\left\{E^*_{1111} + \frac{E_{1122}^2}{E_{1111}} - \left(E^*_{1122} + \frac{E_{1122}^2}{E_{1111}}\right)\right\}$$

giving: $E^*_{1212} = \frac{1}{2}\left(E^*_{1111} - E^*_{1122}\right)$

<u>again</u>, only 2 independent constants

Using these facts in Equations (20 a-e)

$$\boxed{\begin{aligned} E_L = \bar{E} &= \bar{E}^*_{1111} - \frac{(E^*_{1122})^2}{E^*_{1111}} \\ E_T = E &= E^*_{1111} - \frac{(E^*_{1122})^2}{E^*_{1111}} \\ &\vdots \end{aligned}}$$

the same

$$\boxed{\begin{aligned} \nu_{LT} &= \nu = E^*_{1122}/E^*_{1111} \\ \nu_{TL} &= \nu = E^*_{1122}/E^*_{1111} \\ G_{LT} &= \frac{E^*_{1111} - E^*_{1122}}{2} \end{aligned}}$$

the same

If one goes the other way and uses equations (21), then:

$$\boxed{\begin{aligned} E^*_{1111} &= \frac{E}{1-\nu^2} \\ E^*_{1122} &= \frac{\nu E}{1-\nu^2} \\ E^*_{2222} &= \frac{E}{1-\nu^2} \\ E^*_{1212} &= G = \frac{E}{2(1+\nu)} \end{aligned}}$$

← the same as E^*_{1111}

Does $E^*_{1212} = \frac{1}{2}(E^*_{1111} - E^*_{1122})$? check .....

$$E^*_{1212} = \frac{E}{2(1+\nu)} \overset{?}{=} \frac{1}{2}(E^*_{1111} - E^*_{1122}) = \frac{E - \nu E}{2(1-\nu^2)}$$

$$1-\nu^2 = (1+\nu)(1-\nu)$$

$$\Rightarrow \quad \frac{E - \nu E}{2(1-\nu^2)} = \frac{E(1-\nu)}{2(1+\nu)(1-\nu)}$$

So: $$\frac{E}{2(1+\nu)} \stackrel{?}{=} \frac{E}{2(1+\nu)} \qquad \underline{\underline{\text{YES!}}}$$