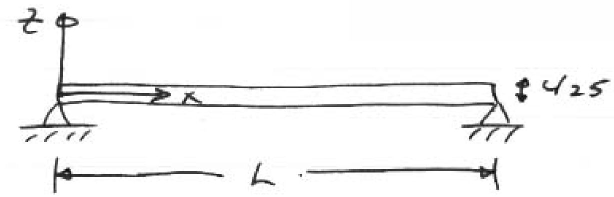
Application Tasks

6.



Steel data: $E = 29 \times 10^6$ psi (Msi)

$\nu = 0.3$

$\alpha = 6.5$ μstrain/°F

given $\Delta T = Ax(3x^2 - 2L^2)$

a) The rigid pins constrain the bar from moving due to temperature differentials (or for any reason). This means that the total elongation will be zero. The total strain is the sum of the thermal and mechanical strains:

$$\underbrace{\epsilon_x = \epsilon_x^T}_{\text{thermal strain}} + \epsilon_x^M \leftarrow \text{mechanical strain}$$

and $\epsilon_x^T = \alpha \Delta T$

$\epsilon_x^M = \sigma_x / E$

$$\Rightarrow \quad \epsilon_x = \alpha \Delta T + \frac{\sigma_x}{E} \qquad (1)$$

The total elongation must be zero. This is the integral of the total strain along the bar:

total elongation $= \int_0^L \epsilon_x \, dx = \int_0^L \left(\alpha \Delta T + \frac{\sigma_x}{E}\right) dx = 0$ (2)

using equation (1)

$$\Rightarrow \int_0^L \frac{\sigma_x}{E} \, dx = -\int_0^L \alpha \Delta T \, dx \qquad (3)$$

σ_x must be a constant since the area is constant and the load must be constant. Use this and the expression for ΔT and progress:

$$\frac{\sigma_x L}{E} = -\alpha \int_0^L A x (3x^2 - 2L^2) \, dx$$

$$= -\alpha A \int_0^L (3x^3 - 2L^2 x) \, dx$$

$$= -\alpha A \left\{ \frac{3x^4}{4} - \frac{2L^2 x^2}{2} \right) \Big]_0^L$$

$$= -\alpha A \left(\frac{3L^4}{4} - L^4 \right)$$

$$\Rightarrow \quad \frac{\sigma_x L}{E} = \frac{1}{4} \alpha A L^4$$

$$\Rightarrow \boxed{\sigma_x = \frac{\alpha A L^3 E}{4}}$$ a constant ✓

Check units

$$\left[\frac{F}{L^2} \right] = \left[\frac{1}{T} \cdot \frac{T}{L^3} \cdot L^3 \cdot \frac{F}{L^2} \right] ✓$$

from part (b)

Use in equation (1) to get:

$$\epsilon_x = \alpha A x (3x^2 - 2L^2) + \frac{1}{4}\alpha A L^3$$

$$\Rightarrow \boxed{\epsilon_x = \alpha A \left(3x^3 - 2L^2 x + \frac{1}{4}L^3\right)}$$

b) use the values for α, E and we are given

$L = 6$ feet

$A = 0.2\ ^{o}F/ft^3$

$$\Rightarrow \epsilon_x = \left(6.5\times10^{-6}\frac{1}{^{o}F}\right)\left(0.2\frac{^{o}F}{ft^3}\right)\left(3x^3 - 72x + 54\right)ft^3$$

(x in [ft])

total: $\boxed{\epsilon_x = (3.9x^3 - 93.6x + 70.2)\times10^{-6} \qquad x \text{ in } [ft]}$

$$\epsilon_x^T = \alpha A x (3x^2 - 2L^2)$$
$$= \left(6.5\times10^{-6}\frac{1}{^{o}F}\right)\left(0.2\frac{^{o}F}{ft^3}\right)\left(3x^3 - 72x\right)ft^3$$

x in [ft]

$$\Rightarrow \boxed{\epsilon_x^T = (3.9x^3 - 93.6x)\times10^{-6} \qquad x \text{ in } [ft]}$$

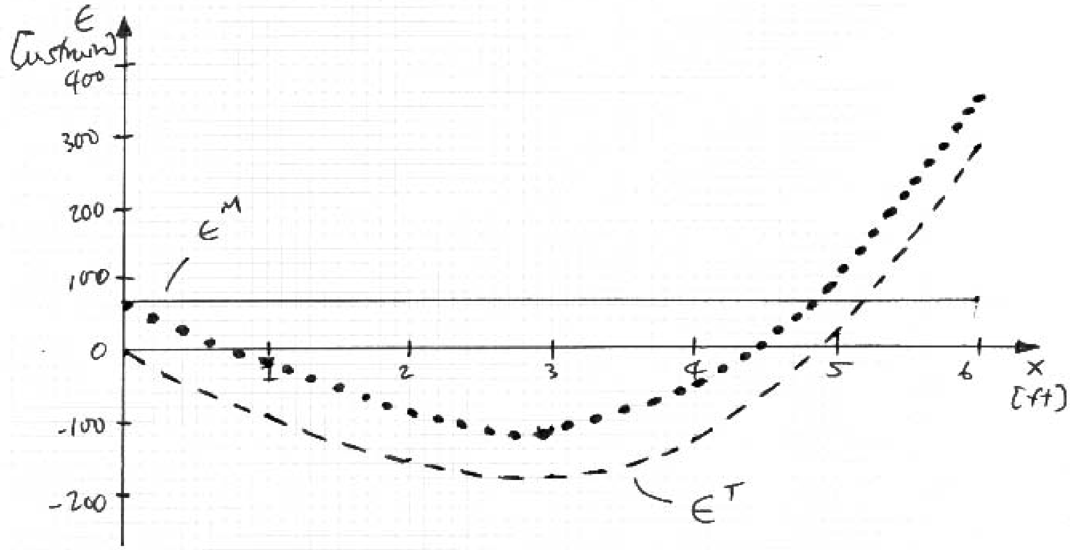
and mechanical:

$$\epsilon_x^M = \sigma_x/E = \frac{1}{4}\alpha A L^3$$
$$= \frac{1}{4}\left(6.5\times10^{-6}\frac{1}{^{o}F}\right)\left(0.2\frac{^{o}F}{ft^3}\right)\left(216 ft^3\right)$$

$$\Rightarrow \boxed{\epsilon_x^M = 70.2\times10^{-6}}$$

Make a table and plot:

x [ft]	ϵ_x^T [$\times 10^{-6}$]	ϵ_x^M [$\times 10^{-6}$]	ϵ_x [$\times 10^{-6}$]
0	0	70.2	70.2
1	-89.7	70.2	-19.5
2	-156	70.2	-85.8
3	-176	70.2	-105
4	-125	70.2	-54.6
5	19.5	70.2	89.7
6	281	70.2	351



c) We have already described the strains in the x-direction. To describe the general state of strain, we need to define ϵ_y and ϵ_z and any shear strains. For isotropic materials (which we have here):

$$\underset{\text{total}}{\epsilon_y} = \epsilon_y^T + \epsilon_y^M$$

$$= (\alpha \Delta T) + \left(\frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E}\right)$$

Have $\sigma_y = 0$ since there is no restriction in the y-direction <u>away</u> <u>from</u> <u>the</u> <u>edge</u>

$$\Rightarrow \epsilon_y = -\nu \frac{\sigma_x}{E} + \alpha \Delta T$$

giving:

$$\epsilon_y = -\nu \frac{\alpha A L^3}{4} + \alpha A x (3x^2 - 2L^2)$$

In a similar manner, can find:

$$\epsilon_z = -\nu \frac{\alpha A L^3}{4} + \alpha A x (3x^2 - 2L^2)$$

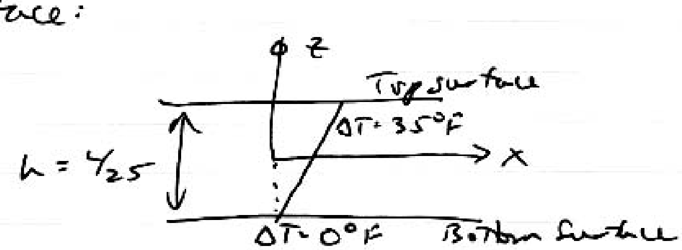
There are no applied shear stresses and isotropic materials show no extension-shear coupling and no thermal strains. Thus:

$$\boxed{\text{all shear strains} = 0}$$

and using our findings:

$$\boxed{\begin{aligned} \epsilon_x &= \alpha A\left(3x^3 - 2L^2x + \tfrac{1}{4}L^3\right) \\ \epsilon_y &= \alpha A\left(3x^3 - 2L^2x - \tfrac{\nu}{4}L^3\right) \\ \epsilon_z &= \alpha A\left(3x^3 - 2L^2x - \tfrac{\nu}{4}L^3\right) \end{aligned}}$$

d) If there is a constant temperature differential between the top and bottom surfaces, bending will occur. The bending is expected to occur because the thermal differential causes more expansion near the top surface of the bar than the bottom surface:



Since all the analyses are linear, we can obtain the stress and strain states due to this temperature differential and then superpose the results with the stress and strain states obtained in part(b) to get the complete solution.

The solution to this through-thickness thermal differential can be found by breaking the problem up into two parts:

(1) Constant temperature thickness through the thickness of 17.5°F

(2) Linearly varying temperature through the thickness

$$\Delta T = \frac{17.5\,°F}{h/2} \cdot z \qquad \text{where } h = \frac{L}{25}$$

$$= \frac{6'}{25}$$

$$\Rightarrow \Delta T = \frac{17.5\,°F}{0.12\,ft}\, z \qquad = 0.24\ ft$$

gives: $\Delta T = (145\ °F/ft)\, z$ $\qquad z$ in [ft]

Part (1): The total strain due to the thermal differential is:

$$\epsilon_x = \epsilon_x^T + \epsilon_x^M$$

$$= \alpha \Delta T + \sigma_x / E$$

where $\Delta T = 17.5\,°F$.

The total elongation of the bar is zero (see equation (2)) and thus:

$$\int_0^L \epsilon_x dx = \int_0^L \left(\alpha \Delta T + \frac{\sigma_x}{E}\right) dx = 0 \qquad (4)$$

$$\Rightarrow \frac{\sigma_x}{E} = -\alpha \Delta T$$

since σ_x is a constant for this case:

$$\Rightarrow \sigma_x = -\alpha E \Delta T$$

$$= -\left(6.5 \times 10^{-6} \frac{1}{°F}\right)\left(29 \times 10^6 psi\right)\left(17.5\,°F\right)$$

$$\Rightarrow \sigma_x = -3299 \text{ psi} \qquad (5)$$

From equation (4) the total strain is zero

$$\Rightarrow \epsilon_x = 0 \qquad (6)$$

and $\epsilon_x^T = \alpha \Delta T = (6.5\times10^{-6})(17.5)$

$$\Rightarrow \epsilon_x^T = 114\times10^{-6} \qquad (7)$$

Part (2): The total strain due to the thermal differential is:

$$\epsilon_x = \alpha \Delta T + \frac{\sigma_x}{E}$$

$$= \alpha (145\ ^\circ F/ft)\, z + \frac{\sigma_x}{E}$$

Again using the condition that the total elongation of the bar is zero:

$$\int_0^L \left(\alpha\, [145\ ^\circ F/ft]\, z + \frac{\sigma_x}{E}\right) dx = 0 \qquad (8)$$

$$\Rightarrow \frac{\sigma_x}{E} = -\alpha \left(145 \frac{^\circ F}{ft}\right) z$$

$$\Rightarrow \sigma_x = -E\alpha \left(145 \frac{^\circ F}{ft}\right) z$$

Note: This is similar to simple beam bending with stress linear through the thickness

This gives: $\sigma_x = (-29\times10^6\,psi)(6.5\times10^{-6}\,\frac{1}{°F})(145\,\frac{°F}{ft})z$

$\Rightarrow \sigma_x = -27330\,psi\,(z)$ $\quad z$ in [ft] (9)

And: $\epsilon_x^T = \alpha\Delta T = 6.5\times10^{-6}\,\frac{1}{°F}\,(145\,\frac{°F}{ft})z$

$\Rightarrow \epsilon_x^T = 943\times10^{-6}\,z$ $\quad$ (z in [ft])

and the total strain is zero

Now get the complete stress and strain states by adding up all the components:

from (6): $\sigma_x = E\epsilon_x^{\text{rt}} = (29\times10^6\,psi)(70.2\times10^{-6})$
$= 2036\,psi$

So: $\sigma_x = \underbrace{2036\,psi}_{\text{from part (5)}} + \underbrace{(-3299\,psi)}_{\text{from constant } \Delta T} + \underbrace{(-27330\,psi)z}_{\text{from linear } \Delta T}$ $\quad z$ in [ft]

$$\Rightarrow \boxed{\sigma_x = -1263\,psi - (27330\,psi)z}$$

z in [ft]

Total strain is:

$\epsilon_x = ([3.9x^3 - 93.6x + 70.2]\times10^{-6}) + (0) + (0)$

(x in [ft])

$$\Rightarrow \boxed{\epsilon_x = (3.9x^3 - 93.6x + 70.2)\times 10^{-6}}$$

(x in [ft])

$$\boxed{\epsilon_{x}^{T} = \underbrace{(3.9x^3 - 93.6x)\times 10^{-6}}_{\text{from (5)}} + \underbrace{114\times 10^{-6}}_{\text{from constant } \Delta T} + \underbrace{943\times 10^{-6}\, z}_{\text{from linear } \Delta T}}$$

x, z in [ft]

and since $\epsilon_x^{M} = \dfrac{\sigma_x}{E}$

$$\Rightarrow \epsilon_x^{M} = \frac{1}{29\times 10^{6}\,\text{psi}}\left(-1263\,\text{psi} - 27330\,\text{psi}\ z\right)$$

$$\Rightarrow \boxed{\epsilon_x^{M} = (-43.6 - 942\, z)\times 10^{-6}}$$

z in [ft]