Application Tasks

6. (a) Let's start with the 3-D orthotropic stress-strain relations:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_1 & -\nu_{13}/E_1 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{23}/E_2 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2G_{12} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

— (1)

For a plane stress state:

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \qquad (2)$$

Using (2) in (1) gives:

$$\epsilon_{11} = 1/E_1\ \sigma_{11} - \nu_{12}/E_1\ \sigma_{22} \qquad (3)$$

$$\epsilon_{22} = -\nu_{12}/E_1\ \sigma_{11} + 1/E_2\ \sigma_{22} \qquad (4)$$

$$\epsilon_{33} = -\nu_{13}/E_1\ \sigma_{11} - \nu_{23}/E_2\ \sigma_{22} \qquad (5)$$

$$\epsilon_{23} = 0 \qquad (6)$$

$$\epsilon_{13} = 0 \qquad (7)$$

$$\epsilon_{12} = 1/2G_{12}\ \sigma_{12} \qquad (8)$$

The plane stress elasticity tensor can be written as:

$$\sigma_{\alpha\beta} = E^{*}_{\alpha\beta\sigma\gamma}\, \epsilon_{\sigma\gamma} \qquad (9)$$

Since the greek subscripts indicate 1, 2, equation (9) relates σ_{11}, σ_{22} and σ_{12} to ϵ_{11}, ϵ_{22} and ϵ_{12}. Thus, we only need to consider equations (3), (4), and (8). In matrix form, these equations can be expressed as:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_1 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/2G_{12} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \qquad (10)$$

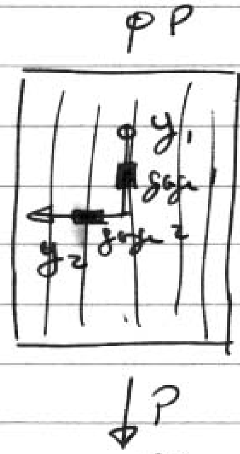
*Note: This is the matrix of the plane stress compliance tensor. The inverse of this is the elasticity tensor.

Thus, from equation (10), we need to know:

$$\boxed{E_1 \quad E_2 \quad \nu_{12} \quad G_{12}}$$

(b) From part (a), there are four material properties that need to be determined. They are determined from the two tests using the stress-strain behavior, assuming that is within the linear region.

Test A



Applied Load/Stress

$\sigma_{11} = 450\ MPa$

$\sigma_{22} = 0$

$\sigma_{12} = 0$

Strain readings

(Gage 1) $\epsilon_{11} = 2600 \times 10^{-6}$

(Gage 2) $\epsilon_{22} = -850 \times 10^{-6}$

Using the known stress and strains in equation (10) gives:

$$\epsilon_{11} = \frac{1}{E_1}(450\ MPa) = 2600 \times 10^{-6}$$

$$\Rightarrow E_1 = \frac{450 \times 10^6\ Pa}{2600 \times 10^{-6}}$$

$$\Rightarrow E_1 = 173 \times 10^9\ Pa \quad \Rightarrow \boxed{E_1 = 173\ GPa}$$

$$\epsilon_{22} = -\frac{\nu_{12}}{E_1}(450\ MPa) = -850 \times 10^{-6}$$

$$\Rightarrow \nu_{12} = 850 \times 10^{-6} \times \frac{173 \times 10^9\ Pa}{450 \times 10^6\ Pa}$$

$$\Rightarrow \boxed{\nu_{12} = 0.327}$$

*Note: Since the load/stress in this case is applied along the material principal axis, E_1 and ν_{12} can be obtained directly from definitions:

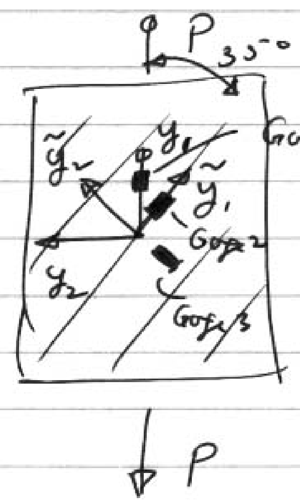
→ Definition of ν_{12}: $\nu_{12} = -\frac{\epsilon_{22}}{\epsilon_{11}}$ (for σ_{11} applied only)

$$\Rightarrow \nu_{12} = \frac{-(-850\times10^{-6})}{2600\times10^{-6}} = 0.327 \checkmark$$

→ Definition of E_1: $E_1 = \frac{\sigma_{11}}{\epsilon_{11}}$ (for σ_{11} applied only)

$$\Rightarrow E_1 = \frac{450\times10^{6}\,Pa}{2600\times10^{-6}} = 173\,GPa \checkmark$$

Test B



We need to set up two sets of axes. Place one aligned with the applied load. Place the other aligned with the fibers (and other two strain gages). We will need to do transformations between the two.

With these axis systems, we summarize what we know:

Applied Load/Stress: $\sigma_{11} = 200\ MPa$

$\sigma_{22} = 0$

$\sigma_{12} = 0$

Measured strains: $\epsilon_{11} = 11900 \times 10^{-6}$ (Gage 1)

(Gage 2) $\tilde{\epsilon}_{11} = 650 \times 10^{-6}$

(Gage 3) $\tilde{\epsilon}_{22} = 10100 \times 10^{-6}$

Now we need to transform $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\sigma}$ in the loading axes to $\tilde{\underset{\sim}{\epsilon}}$ and $\tilde{\underset{\sim}{\sigma}}$ in the material axes in order to find the unknown quantities (as needed):

$$\epsilon_{22},\ \epsilon_{12},\ \tilde{\sigma}_{11},\ \tilde{\sigma}_{22},\ \tilde{\sigma}_{12},\ \tilde{\epsilon}_{12}$$

and to use the stress-strain relations in material axes (as expressed in equation (10)) and rewritten here for the "transformed axis" case:

$$\begin{Bmatrix} \tilde{\epsilon}_{11} \\ \tilde{\epsilon}_{22} \\ \tilde{\epsilon}_{12} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_1 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/2G_{12} \end{bmatrix} \begin{Bmatrix} \tilde{\sigma}_{11} \\ \tilde{\sigma}_{22} \\ \tilde{\sigma}_{12} \end{Bmatrix} \qquad (11)$$

The second order transformation are:

$$\tilde{\epsilon}_{\alpha\beta} = l_{\tilde{\alpha}\sigma} l_{\tilde{\beta}\gamma} \epsilon_{\sigma\gamma} \ ; \quad \tilde{\sigma}_{\alpha\beta} = l_{\tilde{\alpha}\sigma} l_{\tilde{\beta}\gamma} \sigma_{\sigma\gamma}$$

And can be written in matrix form as:

$$\begin{Bmatrix}\tilde{\epsilon}_{11}\\ \tilde{\epsilon}_{22}\\ \tilde{\epsilon}_{12}\end{Bmatrix} = \begin{bmatrix}\cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}\begin{Bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{12}\end{Bmatrix} \quad (12)$$

$$\begin{Bmatrix}\tilde{\sigma}_{11}\\ \tilde{\sigma}_{22}\\ \tilde{\sigma}_{12}\end{Bmatrix} = \begin{bmatrix}\cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix} \quad (13)$$

Noting that our transformation from y_α to $\tilde{y}_\alpha$ is clockwise, we use $\theta = -35^\circ$, so we get:

$$\begin{Bmatrix}\tilde{\sigma}_{11}\\ \tilde{\sigma}_{22}\\ \tilde{\sigma}_{12}\end{Bmatrix} = \begin{bmatrix}0.671 & 0.329 & 0.940\\ 0.329 & 0.671 & -0.940\\ -0.470 & 0.470 & 0.342\end{bmatrix}\begin{Bmatrix}200\text{ MPa}\\ 0\\ 0\end{Bmatrix}$$

$$\Rightarrow \quad \tilde{\sigma}_{11} = 134.2 \text{ MPa} \quad (14a)$$
$$\tilde{\sigma}_{22} = 65.8 \text{ MPa} \quad (14b)$$
$$\tilde{\sigma}_{12} = -94.0 \text{ MPa} \quad (14c)$$

Similarly for the strain using equation (12):

$$\begin{Bmatrix}650\times10^{-6}\\ 10100\times10^{-6}\\ \epsilon_{12}\end{Bmatrix} = \begin{bmatrix}0.671 & 0.329 & 0.940\\ 0.329 & 0.671 & -0.940\\ -0.470 & 0.470 & 0.342\end{bmatrix}\begin{Bmatrix}11900\times10^{-6}\\ 0\\ 0\end{Bmatrix} \quad (15)$$

$$\Rightarrow \quad 650\times10^{-6} = 7985\times10^{-6} + 0.329\epsilon_{22} + 0.94\epsilon_{12} \quad (16)$$

$$10{,}00\times10^{-6} = 3915\times10^{-6} + 0.671\epsilon_{22} - 0.94\epsilon_{12} \quad (17)$$

$$\tilde{\epsilon}_{12} = -5543\times10^{-6} + 0.47\epsilon_{22} + 0.342\epsilon_{12} \quad (18)$$

Solving for ϵ_{22}, ϵ_{12} and $\tilde{\epsilon}_{12}$ via equations (16), (17), and (18), we first add equations (16) and (17):

$$10750\times10^{-6} = 12000\times10^{-6} + \epsilon_{22}$$

$$\Rightarrow \epsilon_{22} = -1250\times10^{-6} \quad (19a)$$

(NOTE: This is the same as using the invariant sum $\epsilon_{11} + \epsilon_{22}$ [see Problem #3])

Use this result in (16) to get:

$$650\times10^{-6} = 7985\times10^{-6} - 411\times10^{-6} + 0.94\epsilon_{12}$$

$$\Rightarrow 0.94\,\epsilon_{12} = -6924\times10^{-6}$$

$$\Rightarrow \epsilon_{12} = -7366\times10^{-6} \quad (19b)$$

And finally in (18):

$$\tilde{\epsilon}_{12} = -5593\times10^{-6} + 0.47(-1250\times10^{-6}) + 0.342(-7306\times10^{-6})$$

$$\Rightarrow \tilde{\epsilon}_{12} = -8650\times10^{-6} \quad (19c)$$

Now use the results of (19a–c) with the stress-strain equations in transformed axes (11) to get:

$$\tilde{\epsilon}_{11} = 650 \times 10^{-6} = \frac{1}{E_1}(134.2\,MPa) - \frac{\nu_{12}}{E_1}(65.8\,MPa) \quad (2a)$$

We already know E_1 (173 GPa) and ν_{12} (0.327), so we check our results via this equation:

$$\frac{134.2\,MPa}{173\,GPa} - \frac{0.327}{173\,GPa}(65.8\,MPa) \stackrel{?}{=} 650 \times 10^{-6}$$

$$651 \times 10^{-6} \stackrel{?}{=} 650 \times 10^{-6} \checkmark$$

yes

Note: NOT exact but we expect some difference due to difference in testing. Actually, 1×10^{-6} is extremely small for experimental deviation.

The next two parts of the matrix equation give:

$$\tilde{\epsilon}_{22} = 10100 \times 10^{-6} = -\frac{\nu_{12}}{E_1}(134.2\,MPa) + \frac{1}{E_2}(65.8\,MPa)$$

$$\Rightarrow E_2 = \frac{65.8\,MPa}{10100 \times 10^{-6} + \frac{0.327}{173\,GPa}(134.2\,MPa)}$$

$$\Rightarrow \boxed{E_2 = 6.36\,GPa}$$

And:

$$\tilde{\epsilon}_{12} = \frac{1}{2G_{12}}(-94.0\,MPa) = -8650 \times 10^{-6}$$

$$\Rightarrow \boxed{G_{12} = 5.43\,GPa}$$

Summarizing:

$$\boxed{\begin{aligned} E_1 &= 173\ GPa \\ E_2 &= 6.36\ GPa \\ \nu_{12} &= 0.327 \\ G_{12} &= 5.43\ GPa \end{aligned}}$$

(c) Note that we used the transformation equation for $\tilde{\epsilon}_{11}$ not to obtain an elastic constant but to check if the relationship held. This shows that we do have more straingages than needed and we can therefore eliminate one gage. However, we cannot choose <u>any</u> gage. We can eliminate Gage 2 in Test A since ν_{12} can be obtained from Test B via the $\tilde{\epsilon}_{11}$ equation (equation (20)). We cannot, however, eliminate any of the gages in Test B or we would not have enough data to obtain $\tilde{\epsilon}_{12}$ and thus G_{12}.