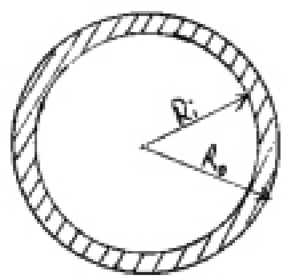
Application Task

1 Consider a tube with outer radius R_o and inner radius R_i.



1) The angle of twist is given by $\alpha = \dfrac{Tl}{GJ}$. In order to compare the applicability of the approximation for thin, closed cross-sections, let α_{approx} be the angle of twist using the approximation and let α_{exact} be the angle of twist using the exact solution. Then, the error in the angle of twist is

$$\alpha_{error} = \frac{\alpha_{exact} - \alpha_{approx}}{\alpha_{exact}} = \frac{\dfrac{Tl}{GJ_{exact}} - \dfrac{Tl}{GJ_{approx}}}{\dfrac{Tl}{GJ_{exact}}}$$

$$\therefore \alpha_{error} = 1 - \frac{J_{exact}}{J_{approx}} \qquad \text{①}$$

The exact value for the torsional constant, J_{exact}, and the approximate value for the torsional constant, J_{approx}, can be obtained from the following equations.

$$J_{exact} = \frac{\pi R_o^4}{2} - \frac{\pi R_i^4}{2} = \frac{\pi}{2}\left[R_o^4 - (R_o - t)^4\right] \quad \text{——— ②}$$

$$J_{approx} = \frac{4A^2}{\oint \frac{ds}{t}} = \frac{4\left[\pi(R_o - \frac{t}{2})^2\right]^2}{2\pi(R_o - \frac{t}{2})\frac{1}{t}} \quad \text{——— ③}$$

$A = \pi(R_o - \frac{t}{2})^2$
$s = 2\pi(R_o - \frac{t}{2})$ } based on midline measurements.

Substituting equation ② and ③ into equation ①, we get,

$$\alpha_{error} = 1 - \frac{\frac{\pi}{2}\left[R_o^4 - (R_o - t)^4\right]}{\frac{4\pi^2(R_o - \frac{t}{2})^4 t}{2\pi(R_o - \frac{t}{2})}}$$

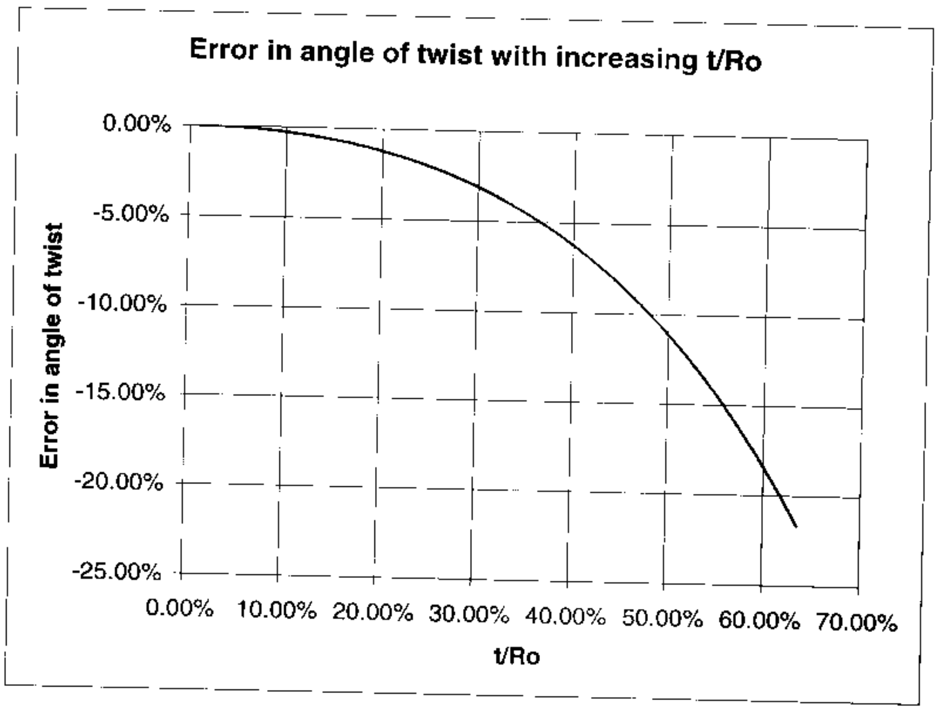
$$= 1 - \frac{\left[R_o^4 - (R_o - t)^4\right]}{(R_o - \frac{t}{2})^3(4t)}$$

$$\alpha_{error} = 1 - \frac{\left[1 - (1 - \frac{t}{R_o})^4\right]}{(1 - \frac{t}{2R_o})^3(4\frac{t}{R_o})} \quad \text{——— ④}$$

Equation ④ shows that for $t = 0$,

$$\alpha_{error} = 1 - \lim_{t \to 0} \frac{\left[1 - (1 - \frac{t}{R_o})^4\right]}{(1 - \frac{t}{2R_o})^3(4\frac{t}{R_o})} = 0$$

As t increases, the error increases. The figure on the next page shows the % error as t increases (as a % of R_o).



b) To check the applicability of the approximation for the maximum shear stress, we will calculate the error, τ_{error}, of the approximate value compared to the exact value.

$$\tau_{error} = \frac{\tau_{exact} - \tau_{approx}}{\tau_{exact}} \quad \text{———} \quad (5)$$

$$\tau_{approx} = \frac{T}{2At} \; , \quad A = \pi\left(R_o - \frac{t}{2}\right)^2 \quad \text{———} \quad (6)$$

↖ constant

$$\tau_{exact} = \frac{T}{J} r \; , \quad J = \frac{\pi}{2}\left(R_o^4 - (R_o - t)^4\right)$$

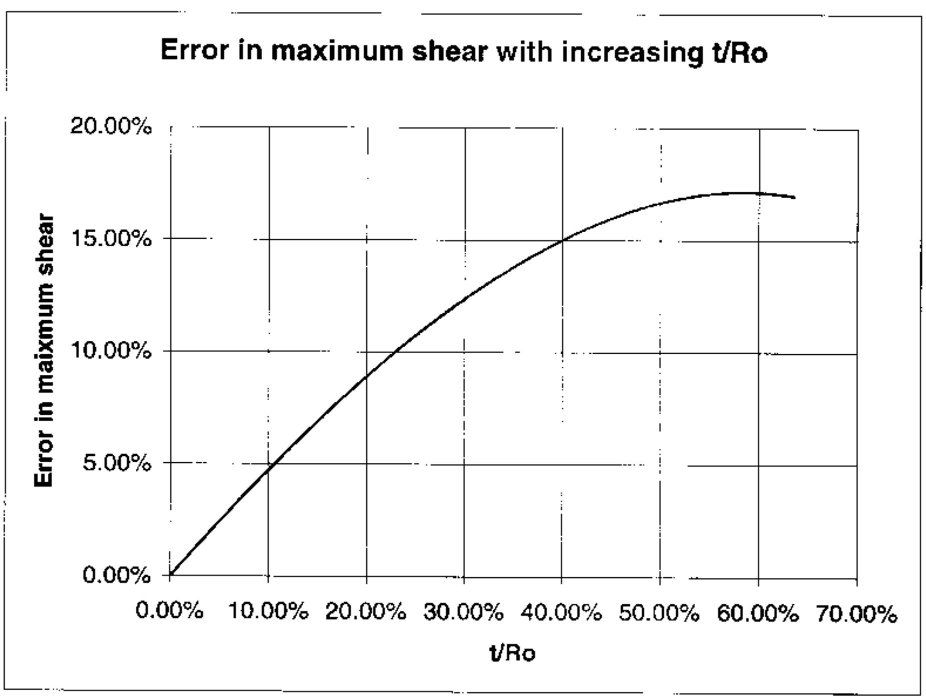
$$\Rightarrow (\tau_{exact})_{max} = \frac{T}{J} R_o . \quad \text{———} \quad (7)$$

Substituting equations ① and ② into equation ⑤, we get,

$$\tau_{error} = \frac{\frac{\cancel{T}R_o}{J} - \frac{\cancel{T}}{2At}}{\frac{TR_o}{J}} = \frac{\frac{R_o}{\frac{\pi}{2}[R_o^4-(R_o-t)^4]} - \frac{1}{2\pi(R_o-\frac{t}{2})^2 t}}{\frac{R_o}{\frac{\pi}{2}[R_o^4-(R_o-t)^4]}}$$

$$\therefore \quad \tau_{error} = 1 - \frac{[R_o^4-(R_o-t)^4]}{4t(R_o-\frac{t}{2})^2 R_o}$$

$$\Rightarrow \quad \tau_{error} = 1 - \frac{[1-(1-\frac{t}{R_o})^4]}{4(\frac{t}{R_o})(1-\frac{t}{2R_o})^2} \qquad \text{⑥}$$



The plot above shows the trend of the e as t/R_o is increased.

From the plots of the error in the angle of twist in part a), and the error in the maximum shear stress in part b), we can find the maximum acceptable thickness ratios, t/R_o, for the two cases if 15% error is considered acceptable.

Angle of twist plot $\Rightarrow$ $t/R_o \leq 56\%$ for $\alpha_{error} \leq 15\%$

Maximum shear $\Rightarrow$ $t/R_o \leq 40\%$ for $\tau_{error} \leq 15\%$

Thus, it is found that a larger thickness is permitted for the twist angle error calculation than for the shear stress error calculation. This is because in the shear stress approximation, we average the actual stress case (see figure below). τ_{error} increases rapidly with thickness because as we increase thickness, we decrease the average τ, while the actual τ_{max} never changes. Note also that τ_{approx} is under predicted whereas α_{approx} is overpredicted.