Unit 5
Engineering Constants

Readings:

Rivello 3.1 - 3.5, 3.9, 3.11

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We do not characterize materials by their E_{mnpq}. The E_{mnpq} are useful in doing transformations, manipulations, etc.

We characterize materials by their

"ENGINEERING CONSTANTS"

(*or*, Elastic Constants)

(what we can physically measure)

There are 5 types

1. Longitudinal (Young's) (Extensional) Modulus: relates extensional strain in the direction of loading to stress in the direction of loading.

   (3 of these)

2. Poisson's Ratio: relates extensional strain in the loading direction to extensional strain in another direction.

   (6 of these…only 3 are independent)

3. Shear Modulus: relates shear strain in the plane of shear loading to that shear stress.
(3 of these)

4. Coefficient of Mutual Influence: relates shear strain due to shear stress in that plane to extensional strain ***or***, relates extensional strain due to extensional stress to shear strain.
(up to 18 of these)

5. Chentsov Coefficient: relates shear strain due to shear stress in that plane to shear strain in another plane.
(6 of these)

Let's be more specific:

1. Longitudinal Modulus
   1) E_{11} or E_{xx} or E_1 or E_x: contribution of ε_{11} to σ_{11}
   2) E_{22} or E_{yy} or E_2 or E_y: contribution of ε_{22} to σ_{22}
   3) E_{33} or E_{zz} or E_3 or E_z: contribution of ε_{33} to σ_{33}

<u>In general</u>: $E_{mm} = \dfrac{\sigma_{mm}}{\varepsilon_{mm}}$ due to σ_{mm} applied <u>*only*</u>

(no summation on m)

2. <u>Poisson's</u> <u>Ratios</u> (***negative*** ratios)

1) ν_{12} or ν_{xy}: (negative of) ratio of ε_{22} to ε_{11} due to σ_{11}
2) ν_{13} or ν_{xz}: (negative of) ratio of ε_{33} to ε_{11} due to σ_{11}
3) ν_{23} or ν_{yz}: (negative of) ratio of ε_{33} to ε_{22} due to σ_{22}
4) ν_{21} or ν_{yx}: (negative of) ratio of ε_{11} to ε_{22} due to σ_{22}
5) ν_{31} or ν_{zx}: (negative of) ratio of ε_{11} to ε_{33} due to σ_{33}
6) ν_{32} or ν_{zy}: (negative of) ratio of ε_{22} to ε_{33} due to σ_{33}

<u>In general</u>: $\nu_{nm} = -\dfrac{\varepsilon_{mm}}{\varepsilon_{nn}}$ due to σ_{nn} applied <u>*only*</u>

(for $n \neq m$)

<u>Important</u>: $\nu_{nm} \neq \nu_{mn}$

However, these are not all independent. There are relations known as "reciprocity relations" (3 of them)

$$\nu_{21} E_{11} = \nu_{12} E_{22}$$

$$\nu_{31} E_{11} = \nu_{13} E_{33}$$

$$\nu_{32} E_{22} = \nu_{23} E_{33}$$

3. Shear Moduli

1) G_{12} or G_{xy} or G_6: contribution of $(2)\varepsilon_{12}$ to σ_{12}
2) G_{13} or G_{xz} or G_5: contribution of $(2)\varepsilon_{13}$ to σ_{13}
3) G_{23} or G_{yz} or G_4: contribution of $(2)\varepsilon_{23}$ to σ_{23}

In general: $G_{mn} = \dfrac{\sigma_{mn}}{2\varepsilon_{mn}}$ due to σ_{mn} applied *only*

factor of 2 here since it relates physical quantities

$$\frac{\text{shear stress}}{\text{shear deformation (angular charge)}} \Rightarrow G_{mn} = \frac{\tau_{mn}}{\gamma_{mn}}$$

4. Coefficients of Mutual Influence (***negative*** ratios)

(also known as "coupling coefficients")

Note: need to use contracted notation here:

1) η_{16}: (negative of) ratio of (2)ε_{12} to ε_{11} due to σ_{11}

2) η_{61}: (negative of) ratio of ε_{11} to (2)ε_{12} due to σ_{12}

3) η_{26} (5) η_{36} (7) η_{14} (9) η_{24}

4) η_{62} (6) η_{63} (8) η_{41} (10) η_{42}

11) η_{34} (13) η_{15} (15) η_{25} (17) η_{35}

12) η_{43} (14) η_{51} (16) η_{52} (18) η_{53}

5. Chentsov Coefficients (***negative*** ratios)

1) η_{46}: (negative of) ratio of (2)ε_{12} to (2)ε_{23} due to σ_{23}
2) η_{64}: (negative of) ratio of (2)ε_{23} to (2)ε_{12} due to σ_{12}
3) η_{45}: (negative of) ratio of (2)ε_{13} to (2)ε_{23} due to σ_{23}
4) η_{54}: (negative of) ratio of (2)ε_{23} to (2)ε_{13} due to σ_{13}
5) η_{56}: (negative of) ratio of (2)ε_{12} to (2)ε_{13} due to σ_{13}
6) η_{65}: (negative of) ratio of (2)ε_{13} to (2)ε_{12} due to σ_{12}

Again, since these are physical ratios, engineering shear strain factor of 2 is used.

Again, these are not all independent. Just as for the Poisson's ratios, there are reciprocity relations. These involve the longitudinal and shear moduli (since these couple extensional and shear or shear to shear). There are 12 of them:

$$\eta_{61} E_1 = \eta_{16} G_6 \qquad \eta_{51} E_1 = \eta_{15} G_5 \qquad \eta_{41} E_1 = \eta_{14} G_4$$

$$\eta_{62} E_2 = \eta_{26} G_6 \qquad \eta_{52} E_2 = \eta_{25} G_5 \qquad \eta_{42} E_2 = \eta_{24} G_4$$

$$\eta_{63} E_3 = \eta_{36} G_6 \qquad \eta_{53} E_3 = \eta_{35} G_5 \qquad \eta_{43} E_3 = \eta_{34} G_4$$

in general:

$$\eta_{nm} E_m = \eta_{mn} G_n$$

$(m = 1, 2, 3)$ **no sum**

$(n = 4, 5, 6)$

and

$$\eta_{46} G_6 = \eta_{64} G_4$$
$$\eta_{45} G_5 = \eta_{54} G_4$$
$$\eta_{56} G_6 = \eta_{65} G_5$$

in general:

$$\eta_{nm} G_m = \eta_{mn} G_n$$

$(m = 4, 5, 6)$ **no sum**

$(m \neq n)$

This gives **21** independent (at most) engineering constants:

Total	$3\ E_n$	$6\nu_{nm}$	$3\ G_m$	$18\eta_{nm}$	$6\ \eta_{nm}$	
	↓	↓	↓	↓	↓	
Indp't.:	3	3	3	9	3	= 21

--> Now that we have defined the terms, we wish to write the "engineering stress-strain equations"

Recall compliances:

$$\varepsilon_{mn} = S_{mnpq}\, \sigma_{pq}$$

and consider only the first equation:

$$\varepsilon_{11} = S_{1111}\, \sigma_{11} + S_{1122}\, \sigma_{22} + S_{1133}\, \sigma_{33} + 2S_{1123}\, \sigma_{23} + 2S_{1113}\, \sigma_{13} + 2S_{1112}\, \sigma_{12}$$

(we'll have to use contracted notation, so…)

$$\varepsilon_1 = S_{11}\, \sigma_1 + S_{12}\, \sigma_2 + S_{13}\, \sigma_3 + S_{14}\, \sigma_4 + S_{15}\, \sigma_5 + S_{16}\, \sigma_6$$

(Note: 2's disappear!)

Consider each of the compliance terms separately:

Case 1: Only σ_{11} applied

$\varepsilon_1 = S_{11} \sigma_1$

and we know

$$E_1 = \frac{\sigma_1}{\varepsilon_1} \quad \text{due to } \sigma_1 \text{ only}$$

$$\Rightarrow \quad S_{11} = \frac{1}{E_1}$$

Case 2: Only σ_{22} applied

$\varepsilon_1 = S_{12} \sigma_2$

We need two steps here.
The direct relation to σ_2 is from ε_2:

$$E_2 = \frac{\sigma_2}{\varepsilon_2} \quad \text{due to } \sigma_2 \text{ only}$$

and we know

$$\nu_{21} = -\frac{\varepsilon_1}{\varepsilon_2} \quad \text{due to } \sigma_2 \text{ only}$$

$$\Rightarrow \quad \frac{\sigma_2}{\varepsilon_1} = -\frac{E_2}{\nu_{21}} \quad \text{due to } \sigma_2 \text{ only}$$

Thus:

$$S_{12} = -\frac{\nu_{21}}{E_2} \quad \text{(due to } \sigma_2 \text{ only)}$$

--> to make this a bit simpler (for later purposes), we recall the reciprocity relation:

$$\nu_{12} E_2 = \nu_{21} E_1$$

$$\Rightarrow \frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$

Thus:

$$S_{12} = -\frac{\nu_{12}}{E_1}$$

<u>Case 3</u>: Only σ_3 applied

In a similar manner we get:

$$S_{13} = -\frac{\nu_{13}}{E_1}$$

<u>Case 4</u>: Only σ_4 (σ_{23}) applied

$$\varepsilon_1 = S_{14}\, \sigma_4$$

Again, two steps are needed.

First the direct relation of ε_4 to σ_4:

$$G_4 = \frac{\sigma_4}{\varepsilon_4} \quad \text{due to } \sigma_4 \text{ only}$$

This is <u>engineering</u> strain!

and then:

$$\eta_{41} = -\frac{\varepsilon_1}{\varepsilon_4} \quad \text{due to } \sigma_4 \text{ only}$$

$$\Rightarrow \frac{\sigma_4}{\varepsilon_1} = -\frac{G_4}{\eta_{41}}$$

$$\Rightarrow S_{14} = -\frac{\eta_{41}}{G_4} \quad \text{due to } \sigma_4 \text{ only}$$

Again, we use a reciprocity relation to get:

$$\frac{\eta_{41}}{G_4} = \frac{\eta_{14}}{E_1}$$

$$\Rightarrow S_{14} = -\frac{\eta_{14}}{E_1} \quad \text{due to } \sigma_4 \text{ only}$$

We do the same for <u>Case 5</u> of only σ_5 applied, and <u>Case 6</u> of only σ_6 applied to get:

$$S_{15} = -\frac{\eta_{15}}{E_1}$$

and

$$S_{16} = -\frac{\eta_{16}}{E_1}$$

With all this we finally get:

$$\varepsilon_1 = \frac{1}{E_1}\left[\sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3 - \eta_{14}\sigma_4 - \eta_{15}\sigma_5 - \eta_{16}\sigma_6\right]$$

we used the reciprocity relations so we could "pull out" this common factor.

We can do this for all the other cases.

In general we write:

$$\varepsilon_n = -\frac{1}{E_n}\sum_{m=1}^{6} \nu_{nm}\,\sigma_m$$

Note: $\nu_{nn} = -1$

If we let η's be ν's

"Engineering" Stress-Strain Equations

General Form

$$\varepsilon_1 = \frac{1}{E_1}\left[\sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3 - \eta_{14}\sigma_4 - \eta_{15}\sigma_5 - \eta_{16}\sigma_6\right]$$

$$\varepsilon_2 = \frac{1}{E_2}\left[-\nu_{21}\sigma_1 + \sigma_2 - \nu_{23}\sigma_3 - \eta_{24}\sigma_4 - \eta_{25}\sigma_5 - \eta_{26}\sigma_6\right]$$

$$\varepsilon_3 = \frac{1}{E_3}\left[-\nu_{31}\sigma_1 - \nu_{32}\sigma_2 + \sigma_3 - \eta_{34}\sigma_4 - \eta_{35}\sigma_5 - \eta_{36}\sigma_6\right]$$

$$\gamma_4 = \varepsilon_4 = \frac{1}{G_4}\left[-\eta_{41}\sigma_1 - \eta_{42}\sigma_2 - \eta_{43}\sigma_3 + \sigma_4 - \eta_{45}\sigma_5 - \eta_{46}\sigma_6\right]$$

$$\gamma_5 = \varepsilon_5 = \frac{1}{G_5}\left[-\eta_{51}\sigma_1 - \eta_{52}\sigma_2 - \eta_{53}\sigma_3 - \eta_{54}\sigma_4 + \sigma_5 - \eta_{56}\sigma_6\right]$$

$$\gamma_6 = \varepsilon_6 = \frac{1}{G_6}\left[-\eta_{61}\sigma_1 - \eta_{62}\sigma_2 - \eta_{63}\sigma_3 - \eta_{64}\sigma_4 - \eta_{65}\sigma_5 + \sigma_6\right]$$

In general:

$$\varepsilon_n = -\frac{1}{E_n} \sum_{m=1}^{6} \nu_{nm} \sigma_m$$

<u>Note</u>: $\nu_{nn} = -1$
η's --> ν's

We have developed these for a fully anisotropic material. Again, there are <u>currently</u> no useful engineering materials of this nature. Thus, these would need to be reduced accordingly.

Orthotropic Case:

In material principal axes, there is no coupling between extension and shear and no coupling between planes of shear, so:

$$\text{all } \eta_{mn} = 0$$

Thus, only the following constants remain:

$$\underbrace{\begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix}}_{3} + \underbrace{\begin{matrix} \nu_{12}, \nu_{21} \\ \nu_{13}, \nu_{31} \\ \nu_{23}, \nu_{32} \end{matrix}}_{3} + \underbrace{\begin{matrix} G_{12} \\ G_{13} \\ G_{23} \end{matrix}}_{3} = \underline{9}$$

*(same as E_{mnpq}, **better be!**)*

So the six equations become:

$$\varepsilon_1 = \frac{1}{E_1}\left[\sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3\right]$$

$$\varepsilon_2 = \frac{1}{E_2}\left[- \nu_{21}\sigma_1 + \sigma_2 - \nu_{23}\sigma_3\right]$$

$$\varepsilon_3 = \frac{1}{E_3}\left[-\nu_{31}\sigma_1 - \nu_{32}\sigma_2 + \sigma_3\right]$$

$$\varepsilon_4 = \frac{1}{G_{23}}\sigma_4$$

$$\varepsilon_5 = \frac{1}{G_{13}}\sigma_5$$

$$\varepsilon_6 = \frac{1}{G_{12}}\sigma_6$$

matrix form:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{S} \; \underset{\sim}{\sigma}$$

this is, in fact, the compliance matrix

Thus:

- If we know the engineering constants (through tests -- this is upcoming)
- Relate engineering constants to S_{mnpq}
- Get E_{mnpq} by inversion of S_{mnpq} matrix (combine steps to directly get relationships between E_{mnpq} and the engineering constants)

Isotropic Case

As we noted in the last unit, as we get to materials with less elastic constants (< 9) than an orthotropic material, we no longer have any more zero terms in the elasticity or compliance matrix, but more nonzero terms are related.

For the isotropic case:

- All extensional moduli are the same:

  $$E_1 = E_2 = E_3 = E$$

- All Poisson's ratios are the same:

  $$\nu_{12} = \nu_{21} = \nu_{13} = \nu_{31} = \nu_{23} = \nu_{32} = \nu$$

- All shear moduli are the same:

  $$G_4 = G_5 = G_6 = G$$

- And, there is a relationship between E, ν and G (from Unified):

$$G = \frac{E}{2(1 + \nu)}$$

Thus, there are only <u>2</u> independent constants.

We now have all the relationships to do the manipulations, but we need to measure the basic properties. We must therefore talk about…

<u>Testing</u>

Testing is used for a variety of purposes.

Depending on the purpose, the technique and "care" will vary. The "fidelity" needed in the testing depends on the use.

Basically, apply a load (stress) condition and measure appropriate responses:

- Strain
- Displacements
- Failure

(or maybe vice versa)

Concerns in test specimens

- Boundary conditions and introduction of load
- Stress concentrations
- Achievement of desired stress state
- Cost and ease of use also important (again, depends on use of test)

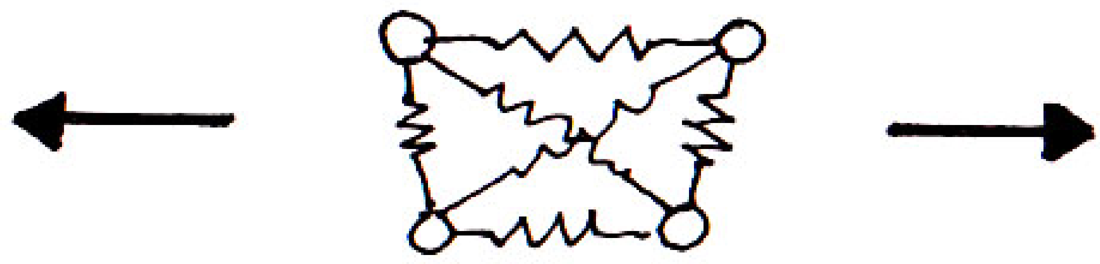
Many of these concerns will depend on the material / configuration and load condition. This generally involves issues of scale.

"Properties" of a material / structure depends on the "scale" at which you look at it.

Scale ≈ level of homogenization (average behavior over a certain size)

<u>Example 1:</u> Atoms make up a material

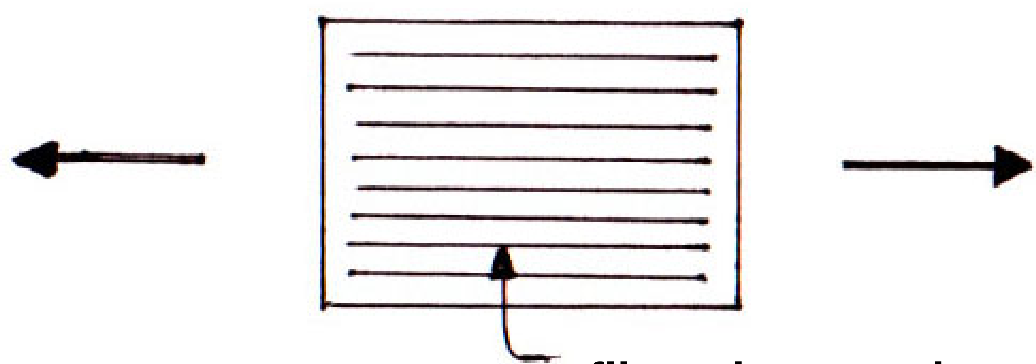
Figure 5.1 **Representation of atomic bonding as springs**



The behavior of the “materials” is some combination of the atoms and their bonds.

<u>Example 2</u>: Composite

Figure 5.2 **Representation of unidirectional composite**

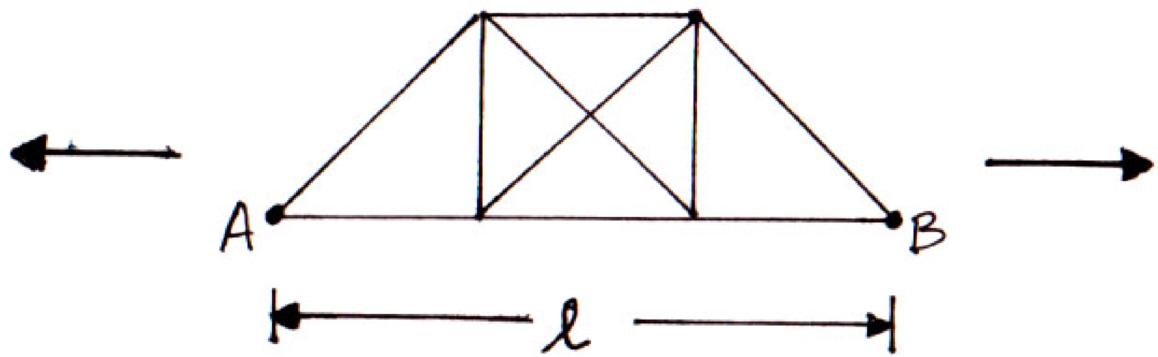


Fibers and matrix respond differently. “Average” their response to get “composite” properties

⇒ Must look at responses at length of several fiber diameters

Example 3: A Truss

Figure 5.3 **Representation of a generic truss**



Each truss member responds in a certain way, but we can characterize the truss behavior on a large scale by looking at the displacement between A and B ($\Delta\ell$) and get a "truss stiffness".

Final Example: put all of these together

- Truss
- Members are composites
- Composites have fibers and matrix
- Fibers and matrix are composed of atoms

⇒ a continuum!

Any material is really a structure. We are making an engineering approximation by characterizing it at a certain level.

$\Rightarrow$ that is the limit / assumption on “material properties” and the tests we use to measure them.

(so elastic constants are an engineering representation of “micromechanical” behavior)

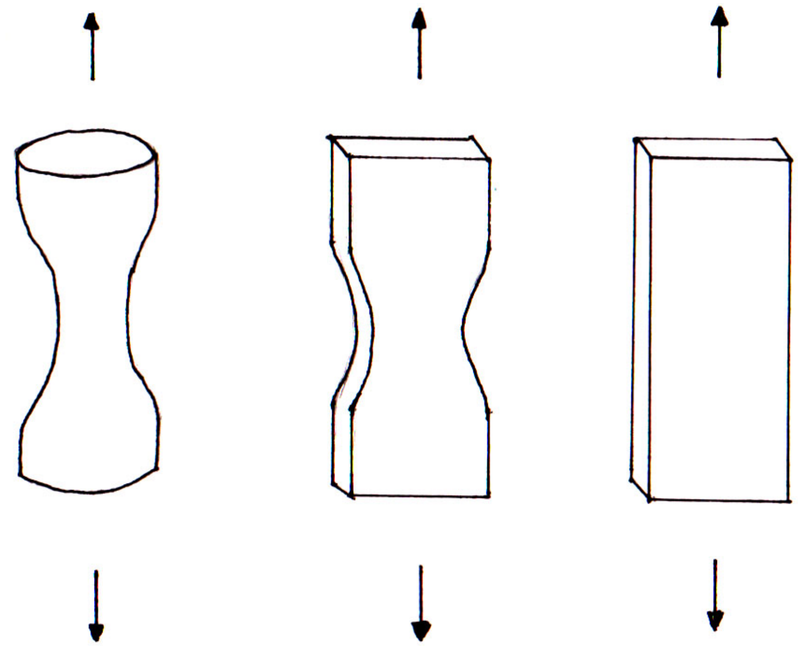
Thus, let’s keep in mind such issues of scale as we consider test methods. Must measure at or above pertinent scale of homogenization / averaging.

There are many different types of tests, but in dealing with elastic constants, there are 3 basic load conditions (tension, compression, shear)

The test specimen will depend on the material and the load condition.

1. <u>Tension</u> (easiest to do)

Figure 5.4 **Tension specimens**



Tapered Bar **Dogbone Specimen** **Straight-edged coupon (composites) …stress concentration problem**

2. <u>Compression</u>

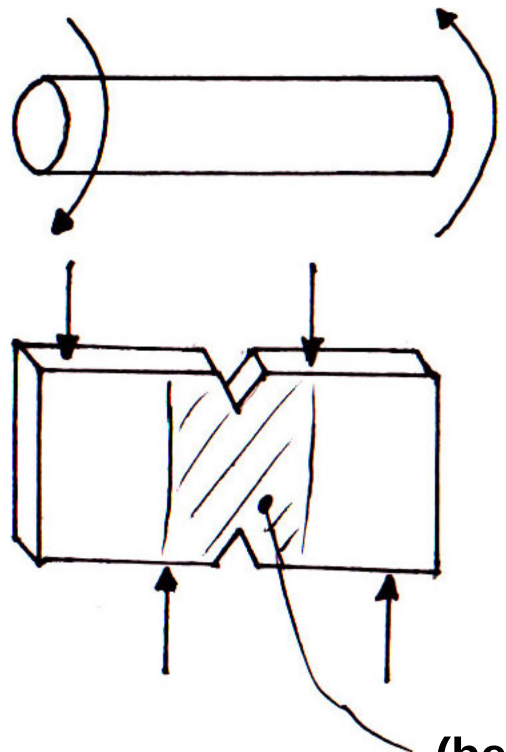
Similar specimens can be used, but must beware of buckling (global and local instabilities)

Possibility of local reinforcement to prevent buckling.

3. <u>Shear</u>

Very hard to apply pure shear.

Figure 5.5 **Possible shear specimens**



Tube

Iosipescu

(beam theory shows area of pure shear -- test section)

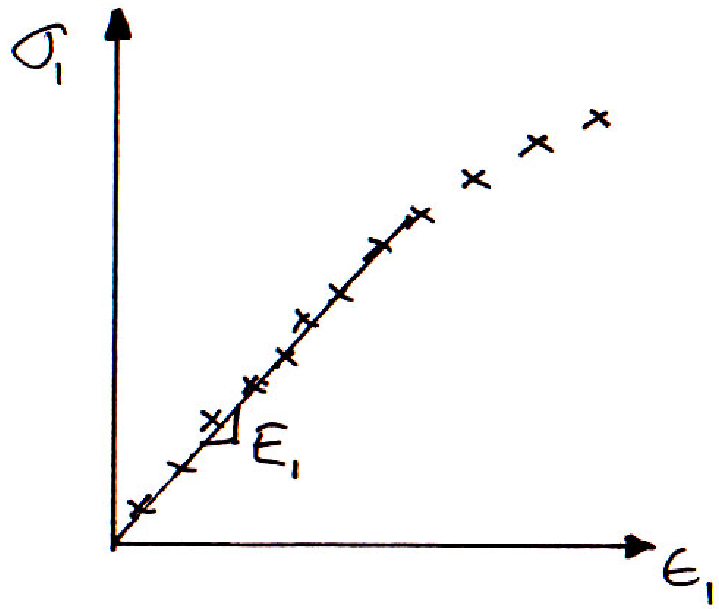
Refer to: ASTM (American Society for Testing and Materials) Annual Book of Standards

Voluntary test standards are contained there.

⇒ what to do with test data

Test data must be "reduced" to give the engineering constants.

Figure 5.6 **Typical stress-strain data (for ductile material)**



- The engineering constants are defined (somewhat arbitrary) as various parts of this curve. Generally within the "<u>initial</u> <u>linear</u> region".
- Often use linear regression

Key
⇒ stress-strain relations with engineering constants attained in this manner are valid *only* in the linear region.

Thus, one must report:

- slope (engineering constant)
- region of applicability

Note: we have not dealt with temperature effects. We will consider this later.

Also note: strength / failure properties are much harder to measure.

(recall Unified: average behavior vs. local / weakest link)

We have now developed the general 3-D stress-strain relations. But we often deal with a problem where we can simplify (model as) to a 2-D system. Two important cases to next consider:

Plane Stress

Plane Strain