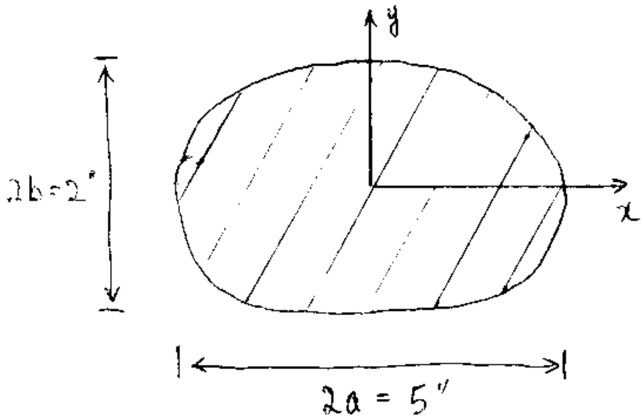
Application Tasks

i.



Given data:

$E = 10.5$ Msi

$\nu = 0.30$

$T = 150$ ft·lbs $= 1800$ in-lbs

$l = 4$ ft $= 48$ in.

Torsion stress function

$$\phi(x,y) = A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

* Note: As required, $\phi = 0$ on contour of ellipse.

a) Find the torsional constant J.

$$J = \frac{T}{GK} \qquad \text{①}$$

applied torque

shear modulus

rate of twist.

We need to find GK and T.

From Poisson's equation,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2GK$$

$$\Rightarrow \quad GK = \frac{1}{2}\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right] \qquad \text{②}$$

The torque, T, is equal to

$$T = -2\iint \phi \, dx dy \qquad \text{③}$$

Plugging ② and ③ into ①, we get

$$T = \frac{-2\iint \phi \, dx dy}{\frac{1}{2}\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right]} \qquad \text{④}$$

Using the stress function, we can evaluate equation ④.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = A\left(\frac{2}{a^2} + \frac{2}{b^2}\right) \qquad \text{⑤}$$

$$\iint \phi \, dx dy = A\iint \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) dx dy = \frac{A}{a^2}\iint x^2 dx dy + \frac{A}{b^2}\iint y^2 dx dy - A\iint dx dy$$

*Note that the integrals, $\iint x^2 dx dy$, $\iint y^2 dx dy$ and $\iint dx dy$ have physical meanings.

$I_y = \iint x^2 dx dy = \frac{\pi}{4}a^3 b$ for an ellipse

$I_x = \iint y^2 dx dy = \frac{\pi}{4}ab^3$ for an ellipse

$\text{Area} = \iint dx dy = \pi ab$ for an ellipse.

So,

$$\iint \phi\, dx\, dy = \frac{A}{a^2}\left(\frac{\pi}{4}a^3 b\right) + \frac{A}{b^2}\left(\frac{\pi}{4}ab^3\right) - A(\pi ab)$$

$$\Rightarrow \quad \iint \phi\, dx\, dy = -\frac{\pi A ab}{2} \qquad \text{⑥}$$

Plugging equations ⑤ and ⑥ into equation ④, we get,

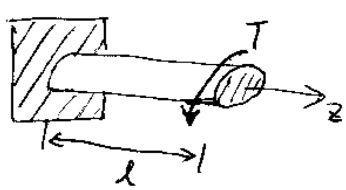
$$J = \frac{-2\left(-\frac{\pi ab}{2}\right)A}{\frac{A}{2}\left(\frac{2}{a^2}+\frac{2}{b^2}\right)} = \frac{\pi ab}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)} = \frac{\pi ab}{\frac{a^2+b^2}{a^2 b^2}}$$

$$\Rightarrow \quad J = \frac{\pi a^3 b^3}{a^2+b^2}$$

Using the given dimensions of $a = 2.5''$ and $b = 1''$, we get

$$\boxed{J = 2.16\pi \text{ in}^4 = 6.77 \text{ in}^4}$$

b) Angle of twist, α, between the two ends.



$$K = \frac{d\alpha}{dz}$$

and

$$T = GJK = GJ\frac{d\alpha}{dz}$$

So,

$$\frac{d\alpha}{dz} = \frac{T}{GJ}$$

$$\Rightarrow \int_0^{\alpha} d\alpha = \int_0^{l} \frac{T}{GJ}\, dz$$

Since T and G are constants, and $J \neq f(z)$,

$$\int_0^{\alpha} d\alpha = \frac{T}{GJ}\int_0^{l} dz$$

$$\Rightarrow \quad \alpha = \frac{Tl}{GJ} \qquad \text{--- (1)}$$

Substituting for J from part a), we get

$$\alpha = \frac{Tl(a^2+b^2)}{G\pi a^3 b^3}$$

Since aluminum is isotropic,

$$G = \frac{E}{2(1+\nu)} = \frac{10.5 \text{ Msi}}{2(1+0.30)} = 4.04 \text{ Msi}$$

Thus,

$$\alpha = \frac{(1800 \text{ in-lbs})(48 \text{ in})(2.5^2+1^2)}{(4.04 \text{ Msi})\,\pi\,(2.5)^3(1)^3}$$

$$\therefore \boxed{\begin{aligned}\alpha &= 3.16\times10^{-3} \text{ rad.}\\ &= 0.18^\circ\end{aligned}}$$

c) Location and value of maximum shear stress.

The resultant shear stress is

$$\tau = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2} \qquad \text{---------- ⑧}$$

The stress components, σ_{xz} and σ_{yz}, are

$$\sigma_{xz} = -\frac{\partial \phi}{\partial y} = -\frac{2Ay}{b^2} \quad , \quad \sigma_{yz} = \frac{\partial \phi}{\partial x} = \frac{2Ax}{a^2}$$

so, plugging into equation ⑧, we get,

$$\tau = \sqrt{\frac{4A^2x^2}{a^4} + \frac{4A^2y^2}{b^4}} \qquad \text{---------- ⑨}$$

To find the maximum value of τ and the point (or points) at which it occurs, we will take its derivative and set it equal to zero. Let's consider $\frac{\partial \tau}{\partial x}$ first.

$$\frac{\partial \tau}{\partial x} = \frac{1}{2}\left[\frac{4A^2x^2}{a^4} + \frac{4A^2y^2}{b^4}\right]^{-1/2}\left[\frac{8Ax}{a^4}\right] = 0$$

$$\therefore \quad x = 0 \text{ is a possibility.}$$

Using $x=0$ to evaluate equation ⑨, we get

$$\tau = \sqrt{\frac{4A^2y^2}{b^4}} = \sigma_{xz}$$

σ_{xz} is largest when y is largest ; $y = \pm b$.

$$\therefore \quad \tau_{max}\Big|_{wrt\ x} = \pm\frac{2A}{b} \quad \text{------} \quad (10)$$

Now, let's consider $\frac{\partial \tau}{\partial y}$.

$$\frac{\partial \tau}{\partial y} = \frac{1}{2}\left[\frac{4A^2x^2}{a^4} + \frac{4A^2y^2}{b^4}\right]^{-\frac{1}{2}}\left[\frac{8A^2y}{b^4}\right] = 0$$

$\therefore \quad y = 0$ is a possibility.

Using $y=0$, we get

$$\tau = \sqrt{\frac{4A^2x^2}{a^4}} = \sigma_{yz}$$

which is largest when x is largest.

$$\therefore \quad \tau_{max}\Big|_{wrt\ y} = \pm\frac{2A}{a} \quad \text{------} \quad (11)$$

Comparing (10) and (11) and noting that $a > b$,

$$\tau_{max} = \sigma_{xz}\Big|_{y=\pm b} = \frac{2A}{b} \quad \text{at} \quad (0, \pm b)$$

To determine the value of A, use the boundary condition.

$$T = -2\iint \phi \, dx\, dy$$

$$\Rightarrow \quad T = \pi A a b = 1800 \text{ in}\cdot\text{lbs}$$

using equation ⑥ from part a)

$$\Rightarrow \quad A = \frac{1800 \text{ in}\cdot\text{lb}}{\pi(2.5'')(1'')}$$

$$\therefore \quad A = 229 \text{ lb/in}$$

Therefore,

$$\tau_{max} = \frac{2\cdot(229 \text{ lb/in})}{1''}$$

$$\Rightarrow \quad \boxed{\tau_{max} = 458 \text{ psi}} \qquad \text{at} \quad (0, \pm b)$$

d). Warping displacement of any section of the rod.

From the class notes on isotropic torsion,

$$\sigma_{yz} = G\left(Kx + \frac{\partial w}{\partial y}\right) \Rightarrow \frac{\partial w}{\partial y} = \frac{\sigma_{yz}}{G} - Kx \quad \text{—— ⑫}$$

$$\sigma_{xz} = G\left(-Ky + \frac{\partial w}{\partial x}\right) \Rightarrow \frac{\partial w}{\partial x} = \frac{\sigma_{xz}}{G} + Ky \quad \text{—— ⑬}$$

From part c), we know σ_{yz} and σ_{xz}. Plugging these into equations ⑫ and ⑬, we get

$$\frac{\partial w}{\partial y} = \frac{2Ax}{Ga^2} - Kx \Rightarrow w_1 = \frac{2Axy}{Ga^2} - Kxy + f_1(x)$$

$$\frac{\partial w}{\partial x} = -\frac{2Ay}{Gb^2} + Ky \quad \Rightarrow \quad w_2 = -\frac{2Axy}{Gb^2} + Kxy + f_2(y)$$

Before comparing w_1 and w_2, rewrite A in terms of a and b.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = A\left[\frac{2}{a^2} + \frac{2}{b^2}\right]$$

and.
$$GK = \frac{1}{2}\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right]$$

$$\therefore \quad GK = A\left[\frac{1}{a^2} + \frac{1}{b^2}\right]$$

$$\Rightarrow \quad A = \frac{GKa^2b^2}{a^2+b^2}$$

Placing A into w's above,

$$w_1 = \left(\frac{2GK\cancel{a^2}b^2}{\cancel{Ga^2}(a^2+b^2)} - K\right)xy + f_1(x) = \left(\frac{2Kb^2}{a^2+b^2} - \frac{K(a^2+b^2)}{a^2+b^2}\right)xy + f_1(x)$$

$$\Rightarrow \quad w_1 = \left(\frac{b^2-a^2}{a^2+b^2}\right)Kxy + f_1(x)$$

$$w_2 = \left(\frac{-2GKa^2b^2}{Gb^2(a^2+b^2)} + K\right)xy + f_2(y) = \left(\frac{-2Ka^2}{a^2+b^2} + \frac{K(a^2+b^2)}{a^2+b^2}\right)xy + f_2(y)$$

$$\Rightarrow \quad w_2 = \left(\frac{b^2-a^2}{a^2+b^2}\right)Kxy + f_2(y)$$

Since $w_1 = w_2$, $f_1(x) = f_2(y)$. Since $f_1(x)$ is a function of x only and $f_2(y)$ is a function of y only, they must both be constant in order for the equality to be satisfied. It is common to choose the constant equal to zero for convenience, so at $x, y = 0$ (center), $w = 0$.

warping function. $w = \left[\frac{b^2 - a^2}{a^2 + b^2}\right] K x y$

Since,

$$K = \frac{T}{GJ} = \frac{1800 \text{ in}\cdot\text{lbs}}{(4.04\text{ Msi})(2.16\pi \text{ in}^4)}$$

$$\Rightarrow \quad K = 6.57 \times 10^{-5} \text{ rad/in}$$

Thus,

$$w = \frac{1^2 - 2.5^2}{2.5^2 + 1^2}\left(6.57\times10^{-5} \text{ rad/in}\right) x y$$

$$\therefore \quad \boxed{w = \left(-4.8\times10^{-5}\right) x y \text{ in}}$$

x, y in inches.

e) Rod is made of steel $\left(\begin{array}{l} E = 30 \text{ Msi} \\ \nu = 0.30 \end{array} \right.$ * Note $\frac{E_{steel}}{E_{Al}} \simeq 3$

a) Torsional constant, J, depends on geometry only.

$$\boxed{J = \text{same.}}$$

b) Angle of twist, α, depends on G $\Rightarrow$ $G = f(E, \nu)$

$$G_{steel} = \frac{E}{2(1+\nu)} = 11.5\ Msi$$

$$\therefore \quad \alpha_{steel} = \alpha_{Al}\left[\frac{G_{Al}}{G_{steel}}\right]$$

$$= \underbrace{\alpha_{Al}}_{0.18^\circ}\left[\frac{4.04\,Msi}{11.5\,Msi}\right]$$

$$\boxed{\alpha_{steel} = 0.06^\circ}$$ down by factor of 3

c) Maximum shear stress, τ_{max}, independent of material

$$\boxed{\tau_{max} = \text{same}}$$

d) Warping function, ω.

Since

$$\omega_{steel} = \frac{T(b^2-a^2)xy}{G\pi a^3 b^3}$$

$$\therefore \quad \omega_{steel} = \omega_{Al}\left[\frac{G_{Al}}{G_{steel}}\right]$$

$$\boxed{\omega_{steel} = (-1.7 \times 10^{-5})\, xy \text{ in}}$$ down by factor of 3.

Similarly, when rod is made of titanium, $\left(\begin{array}{l} E = 15.5 \text{ Msi} \\ \nu = 0.34 \end{array} \right.$

a) $$\boxed{J = \text{same}}$$

b)
$$G_{titanium} = \frac{E}{2(1+\nu)} = 5.78 \text{ Msi}$$

$$\therefore \alpha_{titanium} = \alpha_{Al} \left[\frac{G_{Al}}{G_{titanium}} \right]$$

$$= 0.18° \left[\frac{4.04 \text{ Msi}}{5.78 \text{ Msi}} \right] \approx 0.70$$

$$\boxed{\alpha_{titanium} = 0.13°}$$

c) $$\boxed{\tau_{max} = \text{same}}$$

d)
$$w_{titanium} = w_{Al} \left[\frac{G_{Al}}{G_{titanium}} \right]$$

$$\boxed{w_{titanium} = (-3.4 \times 10^{-5}) xy \text{ in}}$$

f) Torsional rigidity is defined as GJ. So, we need to find a and b (with $\frac{a}{b} = 2.5$) such that the steel rod and the titanium rod have the same GJ as the aluminum rod. Thus,

$$GJ = G_{al} J_{al}$$

$$\Rightarrow \quad J = \frac{G_{al} J_{al}}{G} \qquad \text{———————} \quad (14)$$

From part a), $J = \frac{\pi a^3 b^3}{a^2+b^2}$. Thus, equation (14) becomes

from part a)

$$\frac{\pi a^3 b^3}{a^2+b^2} = \frac{G_{al} J_{al}}{G} = \frac{1}{G}(4.04\,\text{Msi})(6.77\,\text{in}^4)$$

$$\Rightarrow \quad \frac{\pi \left(\frac{a}{b}\right)^2 ab^3}{\left(\frac{a}{b}\right)^2 + 1} = \frac{1}{G}(27.4 \times 10^6\ \text{in}^2 \cdot \text{lb})$$

$\frac{a}{b} = 2.5$

$$\Rightarrow \quad ab^3 = \frac{(27.4 \times 10^6\ \text{in}^2 \cdot \text{lb})\left((2.5)^2+1\right)}{G\,\pi\,(2.5)^2}$$

$$\therefore \quad ab^3 = \frac{1}{G}(10.1 \times 10^6\ \text{in}^2 \cdot \text{lb}) \qquad \text{———————} \quad (15)$$

For steel, $G_{steel} = 11.5\,\text{Msi}$ (from part e)). Using $\frac{a}{b} = 2.5$ and equation (15),

$$2.5\,b^4 = \frac{10.1 \times 10^6\ \text{in}^2\cdot\text{lb}}{11.5\ \text{Msi}}$$

$$\Rightarrow\ b^4 = 0.35\ \text{in}^4$$

$$\therefore\ b = 0.77\ \text{in}$$

$$\Rightarrow\ a = 2.5b = 1.9\ \text{in}$$

Rewriting, for steel,

$$\boxed{a = 1.9\,\text{in},\ \ b = 0.77\,\text{in}}$$

Similarly, for titanium, $G_{titanium} = 5.78\ \text{Msi}$ Using $\frac{a}{b} = 2.5$ and equation (15),

$$2.5\,b^4 = \frac{10.1 \times 10^6\,\text{in}^2\cdot\text{lb}}{5.78\,\text{Msi}}$$

$$\Rightarrow\ b^4 = 0.70\ \text{in}^4$$

$$\therefore\ b = 0.91\ \text{in}$$

$$\Rightarrow\ a = 2.5b = 2.3\,\text{in}$$

$$\Rightarrow\quad \boxed{a = 2.3\,\text{in},\ \ b = 0.91\,\text{in}}$$

Since the maximum shear stress is

$$\tau_{max} = \frac{2A}{b} \quad @ \quad (0, \pm b)$$

and

$$T = \pi A ab \quad \Rightarrow \quad A = \frac{T}{\pi ab}$$

we can rewrite the maximum shear stress as

$$\tau_{max} = \frac{2T}{\pi a b^2} \quad @ \quad (0, \pm b)$$

For steel,

$$\tau_{max} = \frac{2(1800 \text{ in-lbs})}{\pi (1.9)(0.77)^2}$$

$$\Rightarrow \boxed{\tau_{max} = 1020 \text{ psi}} \quad @ \quad (0, \pm 0.77 \text{in})$$

For titanium,

$$\tau_{max} = \frac{2(1800 \text{ in-lbs})}{\pi (2.3)(0.91)^2}$$

$$\Rightarrow \boxed{\tau_{max} = 602 \text{ psi}} \quad @ \quad (0, \pm 0.91 \text{in})$$

4) In evaluating the efficiency and capabilities of the rods made of three materials, it is important to consider the parameters. The first parameter is the torsional rigidity, GJ, defined as

$$GJ = \frac{T}{\underbrace{\frac{d\alpha}{dz}}_{\text{rate of twist}} (= K)}$$

For a rod under an applied torque, T, the greater the torsional rigidity, the smaller the rate of twist, i.e., less deformation. The second parameter is the weight of the rod. In most structural applications, it is desirable to minimize the weight of the structural, everything else being equal. The weight can be computed from

$$W = \rho \pi a b$$

The third parameter is the yield stress of the material. In the case of the rod, we know the maximum resultant shear stress. Thus, using the yield stress, we can find the applied torque required to cause yielding. Since yielding occurs when the maximum shear stress is equal to the yield stress multiplied by some constant (in Tresca condition, $\frac{1}{2}$),

from part f)

$$\tau_{max}\Big|_{yield} = \frac{2\,T_{yield}}{\pi a b^2} \propto \sigma_{yield}$$

$$\Rightarrow \quad T_{yield} \propto \sigma_{yield}\, a b^2$$

A larger T_{yield} indicates that the rod will take more applied torque before yielding.

Let's use the information from a) through f) and the given properties to compute the three parameters.

$$\begin{cases} \rho_{al} = 0.097 \text{ lbs/in}^3 \\ \rho_{steel} = 0.283 \text{ lbs/in}^3 \\ \rho_{titanium} = 0.170 \text{ lbs/in}^3 \end{cases} \qquad \begin{cases} \sigma_{y\,al} = 68 \text{ ksi} \\ \sigma_{y\,steel} = 105 \text{ ksi} \\ \sigma_{y\,titanium} = 172 \text{ ksi} \end{cases}$$

We can get the following table by plugging the given numbers into the equations.

	Torsional Rigidity (lb in^2)	Weight per Unit Length (lb/in)	Yield Torque (lb in)
Aluminum	27.4E+6	0.76	170.0E+3
Steel	26.9E+6	1.30	118.3E+3
Titanium	27.2E+6	1.12	327.6E+3

Note that the torsional rigidity is actually the same and equal to 27.4×10^6 lb in^2. The discrepancies are due to round-off errors. From the table, we can see that the aluminum rod has the smallest weight, while the steel rod has the greatest weight. The yield torque for the steel rod is the lowest while the yield torque for the titanium is the greatest.

Based on the weight and yield torque, the steel rod is the least efficient alternative in terms of torsional capability. The titanium rod has a much greater yield torque than the aluminum rod, but it also weighs about 50% more.

The material variable that is important in evaluating the torsional capability based on the analysis above are the

density
stiffness, and the yield stress.

"shear modulus.