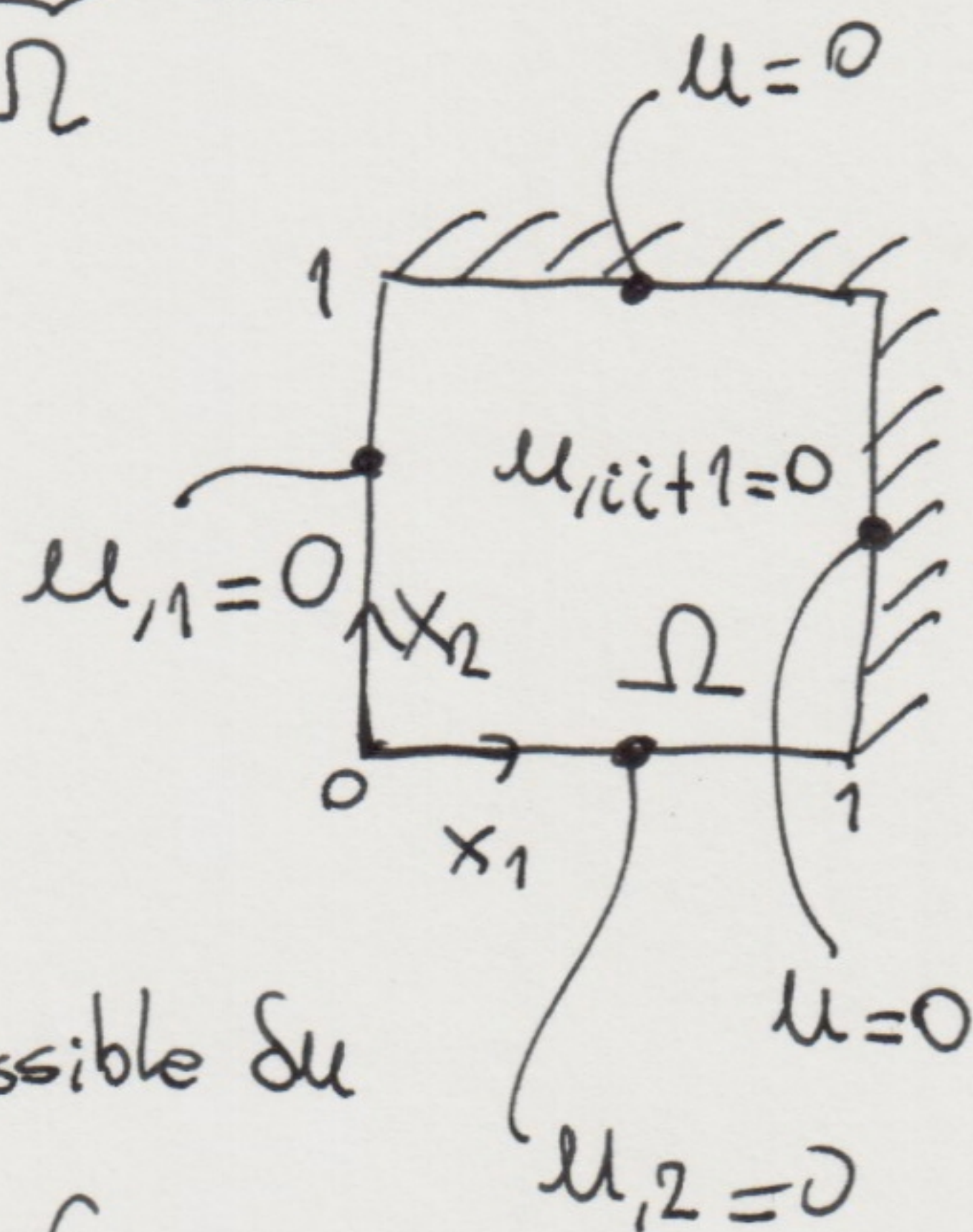**(7.47)**

$$u_{,ii} + 1 = 0 \quad \text{in} \quad \underbrace{[0,1] \times [0,1]}_{\Omega}$$

$$u(1, x_2) = u(x_1, 1) = 0$$

$$u_{,1}(0, x_2) = u_{,2}(x_1, 0) = 0$$

Weak form:

$$\int_\Omega (u_{,ii} + 1) \delta u \, d\Omega = 0$$

$$\text{+ admissible } \delta u$$

$$-\int_\Omega u_{,i} \delta u_{,i} \, d\Omega + \underbrace{\int_\Omega (u_{,i} \delta u)_{,i} \, d\Omega}_{} + \int_\Omega 1 \, \delta u \, d\Omega = 0$$

$$\int_{\partial\Omega} u_{,i} \delta u \, n_i \, d\partial\Omega$$

$$\quad \hookrightarrow \text{ boundary of } \Omega$$

Let's analyze the boundary integral:

$$\int_{\partial\Omega} u_{,i} \delta u \, n_i \, d\partial\Omega = \int_0^1 (u_{,1} \delta u \, n_1 + u_{,2} \delta u \, n_2)\Big|_{(x_1, 0)} dx_1 +$$

$$\int_0^1 (\qquad)\Big|_{(1, x_2)} dx_2 +$$

$$\int_1^0 (\qquad)\Big|_{(x_1, 1)} (-dx_1) +$$

$$\int_1^0 (\qquad)\Big|_{(0, x_2)} (-dx_2)$$

It can be seen that in the essential part of the boundary the integrand vanishes due to the admissibility requirement on δu, i.e., δu must satisfy the homogeneous essential boundary condition:

$$\delta u = 0 \quad \text{where "u" is specified}$$

$$\delta u(1, x_2) = \delta u(x_1, 1) = 0 \quad \text{(boundary)}$$

Therefore, the second and third integrals vanish.

In the first integral, $n_1 = 0$, so the first term $= 0$, $n_2 = -1$ but $u_{,2}(x_1, 0) = 0$, so the integral vanishes.

A similar argument shows that the fourth integral $= 0$

Therefore,

$$\int_{\partial\Omega} u_{,i} \, \delta u \, n_i \, d\partial\Omega = 0$$

The weak form is:

$$\int_{\Omega} u_{,i} \, \delta u_{,i} \, d\Omega = \int_{\Omega} 1 \, \delta u \, d\Omega \qquad \forall \text{ admissible } \delta u$$

This is taken as a basis for approximation:

$$u_{(x_1,x_2)} \sim u_{h_{(x_1,x_2)}} = \phi_{\alpha_{(x_1,x_2)}} u_\alpha \quad , \quad \delta u(x_1, x_2) \sim \delta u_{h}(x_1, x_2) = \phi_\alpha(x_1, x_2) \delta u_\alpha$$

Inserting in weak form:

$$\delta U_a \underbrace{\int_\Omega \phi_{b,i}\, \phi_{a,i}\, d\Omega}_{K_{ab}}\, \underbrace{U_b}_{U_b} = \delta U_a \underbrace{\int_\Omega 1\, \phi_a\, d\Omega}_{R_a} \quad \forall\ \text{admissible } \delta U_a$$

Choose appropriate shape functions and solve system:

$$\phi_a(x_1, x_2) = \cos\left(\frac{\pi}{2}\frac{x_1}{1}\right)\cos\left(\frac{\pi}{2}\frac{x_2}{1}\right)$$

which leads to admissible δu_h, $\forall \delta U_a$, i.e.

$$\phi_a(1, x_2) = \overset{0}{\cos\frac{\pi}{2}}\cos\left(\frac{\pi}{2}\frac{x_2}{1}\right) = 0 \quad \Rightarrow$$

$$\delta u_h(1, x_2) = \phi_1\, \delta U_1 = 0$$

$$\phi_{1,1} = -\frac{\pi}{2}\sin\left(\frac{\pi}{2}x_1\right)\cos\left(\frac{\pi}{2}\frac{x_2}{1}\right)$$

$$\phi_{1,2} = -\frac{\pi}{2}\cos\left(\frac{\pi}{2}x_1\right)\sin\left(\frac{\pi}{2}x_2\right)$$

$$K_{11} = \int_{-1}^{1}\int_{-1}^{1}\left(\phi_{1,1}^2 + \phi_{1,2}^2\right) dx_1\, dx_2$$

solution in Mathematica

$$R_1 = \int_{-1}^{1}\int_{-1}^{1} 1\, \phi_1\, dx_1\, dx_2$$