16.21 Techniques of Structural Analysis and Design
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Unit #2 - Mathematical aside: Vectors, indicial notation and summation convention

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Indicial notation

In 16.21 we'll work in a an euclidean three-dimensional space $\mathbb{R}^3$.

Free index: A subscript index $()_i$ will be denoted a *free index* if it is not repeated in the same additive term where the index appears. *Free* means that the index represents **all** the values in its range.

- Latin indices will range from 1 to, $(i, j, k, ... = 1, 2, 3)$,
- greek indices will range from 1 to 2, $(\alpha, \beta, \gamma, ... = 1, 2)$.

Examples:

1. a_{i1} implies a_{11}, a_{21}, a_{31}. (one free index)
2. $x_\alpha y_\beta$ implies $x_1y_1, x_1y_2, x_2y_1, x_2y_2$ (two free indices).
3. a_{ij} implies $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ (two free indices implies 9 values).
4. $$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0$$

has a free index (i), therefore it represents three equations:

$$\frac{\partial \sigma_{1j}}{\partial x_j} + b_1 = 0$$
$$\frac{\partial \sigma_{2j}}{\partial x_j} + b_2 = 0$$
$$\frac{\partial \sigma_{3j}}{\partial x_j} + b_3 = 0$$

Summation convention: When a *repeated index* is found in an expression (inside an additive term) the summation of the terms ranging all the possible values of the indices is implied, i.e.:

$$\boxed{a_i b_i = \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

Note that the choice of index is immaterial:

$$a_i b_i = a_k b_k$$

Examples:

1. $a_{ii} = a_{11} + a_{22} + a_{33}$
2. $t_i = \sigma_{ij} n_j$ implies the **three** equations (why?):

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$
$$t_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$$
$$t_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$$

Other important rules about indicial notation:

1. An index cannot appear more than twice in a single additive term, it's either free or repeated only once.

$$a_i = b_{ij} c_j d_j \text{ is INCORRECT}$$

2. In an equation the *lhs* and *rhs*, as well as all the terms on both sides must have the same free indices
   - $a_i b_k = c_{ij} d_{kj}$ free indices i, k, CORRECT
   - $a_i b_k = c_{ij} d_{kj} + e_i f_{jj} + g_k p_i q_r$ INCORRECT, second term is missing free index k and third term has extra free index r

Vectors

A *basis* in $\mathbb{R}^3$ is given by any set of linearly independent vectors $\mathbf{e}_i$, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. From now on, we will assume that these basis vectors are orthonormal, i.e., they have a unit length and they are orthogonal with respect to each other. This can be expressed using dot products:

$$\mathbf{e}_1.\mathbf{e}_1 = 1, \mathbf{e}_2.\mathbf{e}_2 = 1, \mathbf{e}_3.\mathbf{e}_3 = 1,$$

$$\mathbf{e}_1.\mathbf{e}_2 = 0, \mathbf{e}_1.\mathbf{e}_3 = 0, \mathbf{e}_2.\mathbf{e}_3 = 0, ...$$

Using indicial notation we can write these expression in very succinct form as follows:

$$\mathbf{e}_i.\mathbf{e}_j = \delta_{ij}$$

In the last expression the symbol δ_{ij} is defined as the *Kronecker delta*:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

Example:

$$\begin{aligned} a_i\delta_{ij} =& a_1\delta_{11} + a_2\delta_{21} + a_3\delta_{31}, \\ & a_1\delta_{12} + a_2\delta_{22} + a_3\delta_{32}, \\ & a_1\delta_{13} + a_2\delta_{23} + a_3\delta_{33} \\ =& a_1 1 + a_2 0 + a_3 0, \\ & a_1 0 + a_2 1 + a_3 0, \\ & a_1 0 + a_2 0 + a_3 \\ =& a_1, \\ & a_2, \\ & a_3 \end{aligned}$$

or more succinctly: $\boxed{a_i\delta_{ij} = a_j}$, i.e., the Kronecker delta can be thought of an "index replacer".

A vector $\mathbf{v}$ will be represented as:

$$\mathbf{v} = v_i\mathbf{e}_i = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$$

The v_i are the *components* of $\mathbf{v}$ in the basis $\mathbf{e}_i$. These components are the projections of the vector on the basis vectors:

$$\mathbf{v} = v_j \mathbf{e}_j$$

Taking the dot product with basis vector $\mathbf{e}_i$:

$$\mathbf{v}.\mathbf{e}_i = v_j(\mathbf{e}_j.\mathbf{e}_i) = v_j \delta_{ji} = v_i$$

Transformation of basis

Given two bases $\mathbf{e}_i, \tilde{\mathbf{e}}_k$ and a vector $\mathbf{v}$ whose components in each of these bases are v_i and $\tilde{v}_k$, respectively, we seek to express the components in basis in terms of the components in the other basis. Since the vector is unique:

$$\mathbf{v} = \tilde{v}_m \tilde{\mathbf{e}}_m = v_n \mathbf{e}_n$$

Taking the dot product with $\tilde{\mathbf{e}}_i$:

$$\mathbf{v}.\tilde{\mathbf{e}}_i = \tilde{v}_m(\tilde{\mathbf{e}}_m.\tilde{\mathbf{e}}_i) = v_n(\mathbf{e}_n.\tilde{\mathbf{e}}_i)$$

But $\tilde{v}_m(\tilde{\mathbf{e}}_m.\tilde{\mathbf{e}}_i) = \tilde{v}_m \delta_{mi} = \tilde{v}_i$ from which we obtain:

$$\boxed{\tilde{v}_i = \mathbf{v}.\tilde{\mathbf{e}}_i = v_j(\mathbf{e_j}.\tilde{\mathbf{e}}_i)}$$

Note that $(\mathbf{e_j}.\tilde{\mathbf{e}}_i)$ are the *direction cosines* of the basis vectors of one basis on the other basis:

$$\mathbf{e}_j.\tilde{\mathbf{e}}_i = \|\mathbf{e}_j\| \|\tilde{\mathbf{e}}_i\| \cos \widehat{\mathbf{e}_j \tilde{\mathbf{e}}_i}$$