L3

* Exercise: Verify Euler-Lagrange equations corresponding to this functional are the field equations of L.E.

$$\frac{\partial F}{\partial u_i} - \left(\frac{\partial F}{\partial u_{i,j}}\right)_{,j} = 0 \quad \text{in } B \qquad (A)$$

$$\frac{\partial \phi}{\partial u_i} - \frac{\partial F}{\partial u_{i,j}} n_j = 0 \quad \text{on } S_2 \qquad (B)$$

$$J(u) = \int_B F(x, u, \nabla u)\, dv - \int_{S_2} \phi(x, u)\, ds$$

$$u = \bar{u} \quad \text{on } S_1$$

For Hu-Washizu:

$$F(u, \varepsilon, \sigma) = W(\varepsilon) - f_i u_i + \sigma_{ij}(u_{(i,j)} - \varepsilon_{ij})$$

$$(A)_u \quad -f_i - \sigma_{ij,j} = 0 \qquad \text{(equil.)}$$

Ⓐ$_\varepsilon$ $\quad \dfrac{\partial W}{\partial \varepsilon_{ij}} - \sigma_{ij} = 0 \qquad$ (const.)

Ⓐ$_\sigma$ $\quad u_{(i,j)} - \varepsilon_{ij} = 0 \qquad$ (compatibility)

Ⓑ$_u$ $\quad \phi = n_j \sigma_{ij} (u_i - \bar{u}_i) \quad$ on $S_1$

$$\frac{\partial \phi}{\partial u_i} - \frac{\partial F}{\partial u_{i,j}} n_j = n_j \sigma_{ij} - n_j \sigma_{ij} = 0$$

$$\phi = \bar{t}_i u_i \quad \text{on } S_2$$

$$\frac{\partial \phi}{\partial u_i} = \bar{t}_i \ , \ \frac{\partial F}{\partial u_{i,j}} n_j = \sigma_{ij} \rightarrow \bar{t}_i = \sigma_{ij} n_j$$

etc.

<u>Specialized (simplified) variational principles</u>

- Assume compatibility: $\qquad u_{(i,j)} = \varepsilon_{ij}$ in $B$
  (want $J(u, \sigma)$) $\qquad u = \bar{u}_i$ on $S_1$

$$J = \int_B \left( W(\varepsilon) - f_i u_i \right) dv - \int_{S_2} \bar{t}_i u_i \, ds$$

Legendre transformation: $\chi(\sigma) = \sigma_{ij}\varepsilon_{ij} - W(\varepsilon)$

$$\boxed{J(u,\sigma) = \int_B \Big(\sigma_{ij}\underbrace{\varepsilon_{ij}}_{u(i,j)} - \chi(\sigma) - f_i u_i\Big)\,dV - \int_{S_2} \bar{t}_i u_i \, dS}$$

→ Hellinger-Reissner principle

Euler → equilibrium + constitutive

- Assume compatibility + equilibrium

$$\left.\begin{aligned} \sigma_{ij,j} + f_i &= 0 \quad \text{in } B \\ \sigma_{ij} n_j &= \bar{t}_i \end{aligned}\right\} + \text{ conditions above}$$

start from Hellinger-Reissner:

$$J(\sigma) = \int_B \cancel{-\sigma_{ij,j} u_i} - \chi(\sigma) \cancel{- f_i u_i}\, dV - \int_{S_2} \bar{t}_i u_i \, dV +$$

$$+ \int_S \sigma_{ij} n_j u_i \, dS$$

equil. assumed

$$= -\int_B \chi(\sigma)\, dV - \int_{S_2} \cancel{\bar{t}_i u_i \, dV} + \int_{S_2} \cancel{\sigma_{ij} n_j u_i \, dS} + \int_{S_1} \sigma_{ij} n_j \bar{u}_i \, dS$$

$$\boxed{J(\sigma) = \int_{S_1} \sigma_{ij} n_j \bar{u}_i \, dV - \int_B \chi(\sigma)\, dV}$$

→ Complementary energy principle

Euler → constitutive

- Assume compatibility, constitutive

$$\varepsilon_{ij} = u_{(i,j)} \quad \text{in } B$$
$$u_i = \bar{u}_i \quad \text{on } S_1$$
$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}(\varepsilon) \quad \text{in } B$$

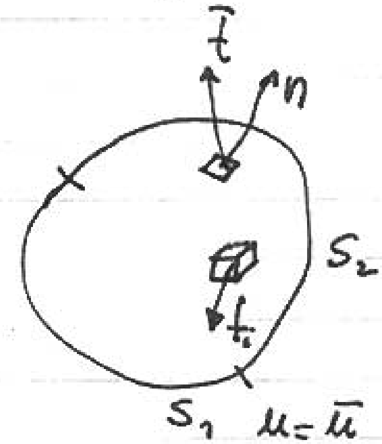
$$\boxed{J(u) = \int_B \left(W(\varepsilon) - f_i u_i\right) dV - \int_{S_2} \bar{t}_i u_i \, dS}$$

→ "Minimum" potential energy principle theorem
Euler → equilibrium

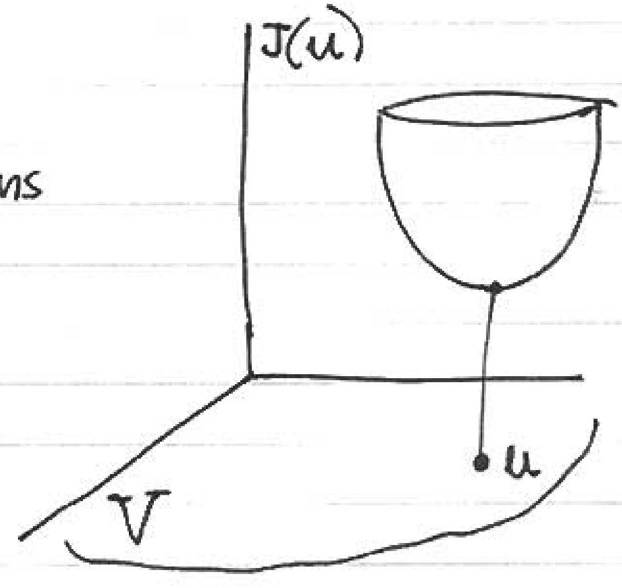
## Approximation theory

Variational principle:

$$J(u) = \int_B \left[\frac{1}{2} C_{ijkl}(x)\, u_{i,j}\, u_{k,l} - f_i u_i\right] dV - \int_{S_2} \bar{t}_i u_i \, ds$$




$$\boxed{J(u) = \inf_{v \in V} J(v)}$$




$V \equiv$ space of functions "$v$" / $v\big|_{S_1} = \bar{u}$

Instead of trying to find exact solution "$u$", try to find "approximate solutions" of the form: $u_h(x) = \sum_{a=1}^{N} \underline{u}_a N_a(x)$

$N_a(x) \equiv$ shape functions

$\underline{u}_a \equiv$ displacement coefficients

$u_h(x) = \bar{u}(x)$ for $x \in S_1$

Three ways of choosing $u_h$

① Rayleigh-Ritz method

② Weighted residuals / Galerkin

③ "best approximation" method

① Rayleigh-Ritz method:

Minimize functional in a sub-space of $V$,

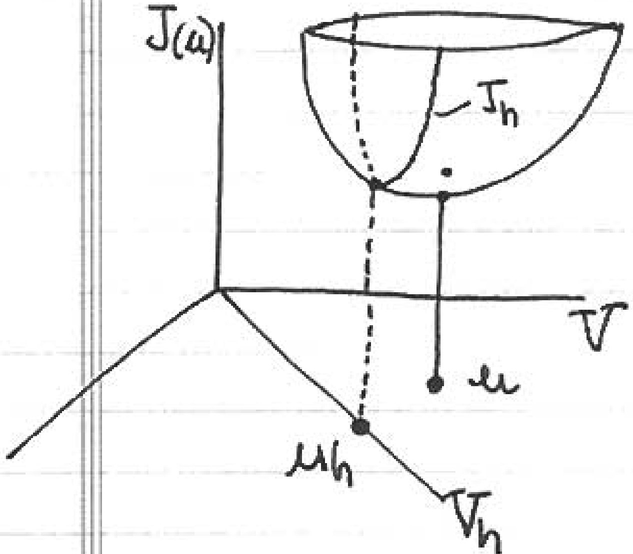
$$V_h = \left\{ u_h = \sum_{i=1}^{n} \underline{u}_a N_a(x) \right\} \quad x \in B \subset \mathbb{R}^d$$

$V_h$ is a finite-dimensional space of dimensions sum because these are not tensors

$n \times d.$

$u_h, v_h \subseteq V_h \qquad u_h = \sum_{a=1}^{n} u_a N_a$

$$v_h = \sum_{a=1}^{n} v_a N_a$$




Obtain $u_h$ by constrained minimization:

constrained potential

$$\boxed{J(u_h) = \min_{v_h \in V_h} J(v_h)}$$

$$J(u_h) = \int_B \left[\frac{1}{2} C_{ijkl} \left(\sum_{a=1}^{n} u_{ia} N_{a,j}\right)\left(\sum_{b=1}^{n} u_{kb} N_{b,l}\right) - \right.$$

$$\left. - f_i \left(\sum_{a=1}^{n} u_{ia} N_a\right)\right] dV - \int_{S_2} t_i \left(\sum_{a=1}^{n} u_{ia} N_a\right) dS$$

$$= \frac{1}{2} \sum_{a=1}^{n} \sum_{b=1}^{n} \underbrace{\left(\int_B C_{ijkl} N_{a,j} N_{b,l}\right) dV}_{K_{iakb}} u_{ia} u_{kb}$$

$$-\sum_{a=1}^{N}\underbrace{\left[\int_B f_i N_a dV + \int_{S_2} \bar{t}_i N_a dS\right]}_{f_{ia}^{ext}} u_{ia}$$

Stiffness matrix: $K_{iakb} = \int_B C_{ijkl} N_{a,k} N_{b,l} dV$

Effective external forces: $f_{ia} = \int_B f_i N_a dV - \int_{S_2} \bar{t}_i N_a dS$

$$\boxed{J(u_h) = \frac{1}{2}\sum_{a=1}^{n}\sum_{b=1}^{n} K_{iakb} u_{ia} u_{kb} - \sum_{a=1}^{n} f_{ia}^{ext} u_{ia} \equiv J_h(u_h)}$$

Depends algebraically on displacement coefficients $u_{ia}$

<u>Minimize:</u>

$$\frac{\partial J_h}{\partial u_{ia}} = 0 \implies \boxed{\sum_{b=1}^{n} K_{iakb} u_{kb} = f_{ia}^{ext}}$$

<u>Indexing</u>; Matrix expressions, how to go from

2 indices to 1.

d: dimension of space (displacements)
(dof_node) 2D: d=2
3D: d=3

$$\underline{u}_a = \begin{Bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{d1} \\ u_{12} \\ \vdots \\ u_{1a} \\ \vdots \\ u_{da} \\ \vdots \\ u_{1n} \\ \vdots \\ u_{dn} \end{Bmatrix}$$

$(d \times n) \times 1$

$(ia) \longrightarrow p$

$p = (a-1) * d + i$

FORTRAN

$\underset{\sim}{K}_{pq}$ $(d \times n) \times (d \times n)$

$p = (a-1) * d + i$
$q = (b-1) * d + k$